

Analytically expressed constraint on two Majorana phases in neutrinoless double beta decay

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Abstract

We assume that neutrinoless double beta decay is caused by the exchange of three light Majorana neutrinos. Under this assumption, we obtain, by the method of perturbation, the equation representing the isocontour of effective Majorana mass which is the function of two CP-violating Majorana phases. The equation representing the isocontour (constraint equation between two Majorana phases) is expressed analytically by six parameters: two lepton mixing angles, two kinds of neutrino mass squared differences, lightest neutrino mass scale, and the effective Majorana mass. We discuss how the constraint equation between two Majorana phases changes when the lightest neutrino mass scale is varied.

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1 Introduction

At present, it is unknown whether the neutrinos are massive Dirac particles or massive Majorana particles. If neutrinoless double beta decay ($0\nu\beta\beta$) is caused by the exchange of three light Majorana neutrinos, the observation of $0\nu\beta\beta$ is the evidence that the neutrinos are Majorana fermions [1, 2, 3]. In the process of neutrinoless double beta decay,

$$(A, Z) \rightarrow (A, Z + 2) + e^- + e^-, \quad (1)$$

the lepton number is not conserved. Neutrinoless double beta decay has not been observed yet. Many experiments of $0\nu\beta\beta$ are in progress and planned: CANDLES [4], NEMO-3 [5], SOLOTVINO [6], CUORICINO [7], EXO-200 [8], KamLAND-Zen [9], etc. If neutrinos are Majorana fermions, lepton mixing matrix (MNS matrix [10]) U is represented by six parameters: three lepton mixing angles ($\theta_{12}, \theta_{23}, \theta_{13}$), CP-violating Dirac phase δ , and two CP-violating Majorana phases α, β . These α and β are the degrees of freedom of phase which come from the assumption that neutrinos are Majorana fermions [11]. Among these six parameters in the lepton mixing matrix, three parameters θ_{12} , θ_{23} , and θ_{13} are measured by experiments [12]. The measurement of θ_{13} especially was done recently [13], whose effect on the study of $0\nu\beta\beta$ had been investigated [14]. Dirac phase δ and two Majorana phases α and β are unknown by experiments. If one assumes that neutrinoless double beta decay is caused by the exchange of three light Majorana neutrinos, the amplitude of $0\nu\beta\beta$ is proportional to the effective Majorana mass $|m_{ee}|$,

$$|m_{ee}| = \left| m_1 |U_{e1}|^2 + m_2 |U_{e2}|^2 e^{2i\alpha} + m_3 |U_{e3}|^2 e^{2i\beta} \right|, \quad (2)$$

where m_1, m_2 , and m_3 are mass of three neutrinos, and U is the lepton mixing matrix. There are two possibilities of neutrino mass spectrum: the normal mass ordering ($m_3 > m_2 > m_1$) and the inverted mass ordering ($m_2 > m_1 > m_3$).

The effective Majorana mass $|m_{ee}|$ depends on the seven parameters: mixing angles (θ_{12}, θ_{13}), neutrino mass (m_1, m_2, m_3), and Majorana phases (β, α). As regarding these seven parameters, mass squared differences $\Delta m_{\odot}^2 \equiv m_2^2 - m_1^2$ and $\Delta m_A^2 \equiv |m_3^2 - m_1^2| \approx |m_3^2 - m_2^2|$, θ_{12} , and θ_{13} are measured by experiments [12], whereas the absolute neutrino mass scale and Majorana phases β, α (if Majorana particles) are not measured. While information on Majorana phases β, α is obtained by neutrinoless double beta decay experiments [15, 16, 17, 18, 19], it is notable that the other experiments measuring the absolute neutrino mass scale are also important to determine the phases β, α as discussed in Ref.[19]. In studying neutrinoless double beta decay, people often use the method in which one regards the lightest neutrino mass as a free parameter and analyzes the effective Majorana mass $|m_{ee}|$ for each given value of the lightest neutrino mass. The effective Majorana mass $|m_{ee}|$ is considered to be a function of two variables β and α provided that the value of the lightest neutrino mass is given by hand. Even if the value of $|m_{ee}|$ is obtained by $0\nu\beta\beta$ experiments, we can not determine both values of β and α simultaneously for a given value of the lightest neutrino mass. Instead,

we can obtain the constraint between the Majorana phases, β and α . Before θ_{13} was measured by the experiments [13], the constraint between β and α had been examined for given values of $|m_{ee}|$ and neutrino mass m_i ($i = 1, 2, 3$) by numerical calculations [20, 16, 17, 21, 23]. In the numerical calculations, the authors gave some values to the parameters, $\theta_{12}, \theta_{13}, \Delta m_{\odot}^2, \Delta m_{\text{A}}^2$, and the lightest neutrino mass, respectively.

In this paper, we obtain the analytic equation representing the isocontour of the effective Majorana mass $|m_{ee}|$ on the $\beta\alpha$ -plane for each of the given values of the lightest neutrino mass. The equation of the isocontour of $|m_{ee}|$ (constraint between β and α) is derived by the method of perturbation for the normal mass ordering case and the inverted mass ordering case, respectively. The equation is represented analytically by the six parameters: $\theta_{12}, \theta_{13}, \Delta m_{\odot}^2, \Delta m_{\text{A}}^2$, the lightest neutrino mass scale, and of course $|m_{ee}|$. Because the effective Majorana mass is invariant under $\beta \rightarrow \beta + n\pi, \alpha \rightarrow \alpha + m\pi$, ($n, m \in \mathbf{Z}$),

$$|m_{ee}|(\beta, \alpha) = |m_{ee}|(\beta + n\pi, \alpha + m\pi), \quad (3)$$

the isocontour of $|m_{ee}|$ around a point (β, α) in the $\beta\alpha$ -plane is the same as that around the point $(\beta + n\pi, \alpha + m\pi)$. If neutrinoless double beta decay is observed, the next challenging task is to restrict the values of the Majorana phases β and/or α . At that stage, the analytically expressed constraint between β and α is useful.

The paper is organized as follows. In section 2, the notations we use are introduced, and the characteristic features of the normal mass ordering and the inverted mass ordering are described, respectively. In section 3, the isocontour of the effective Majorana mass $|m_{ee}|$ in the $\beta\alpha$ -plane (constraint between β and α) is obtained in the case of the normal mass ordering. To describe more concretely, the isocontour of $|m_{ee}|$ around the point of maximum $|m_{ee}|$ in the $\beta\alpha$ -plane, that around the point of minimum $|m_{ee}|$, and that except around the point of maximum or minimum $|m_{ee}|$ are obtained by the method of perturbation. In section 4, the same kind of things as section 3 is described in the case of the inverted mass ordering. Section 5 is devoted to conclusions. In Appendix, the perturbative method used in section 3 is explained.

2 Formalism

If we assume that neutrinoless double beta decay ($0\nu\beta\beta$) is generated by the exchange of the three Majorana neutrinos with light mass, the amplitude of $0\nu\beta\beta$ is proportional to the effective Majorana mass $|m_{ee}|$,

$$|m_{ee}| = \left| \sum_{i=1}^3 m_i U_{ei}^2 \right| = \left| m_1 U_{e1}^2 + m_2 U_{e2}^2 + m_3 U_{e3}^2 \right|, \quad (4)$$

where $m_i (i = 1, 2, 3)$ is neutrino mass of i -th mass eigenstate. The unitary matrix U is the lepton mixing matrix (MNS matrix) and parametrized as follows [14],

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha} & 0 \\ 0 & 0 & e^{i\beta} \end{pmatrix}, \quad (5)$$

where s_{ij} and c_{ij} are sine and cosine of the lepton mixing angle θ_{ij} , respectively. The parameter δ is the CP-violating Dirac phase, and α, β represent the CP-violating Majorana phases. Among nine parameters, (m_1, m_2, m_3) , $(\theta_{12}, \theta_{23}, \theta_{13})$, δ , (β, α) , the following five quantities are known by experiments: i.e., two mass squared differences $\Delta m_{\odot}^2 = m_2^2 - m_1^2$, $\Delta m_{\text{A}}^2 = |m_3^2 - m_1^2| \approx |m_3^2 - m_2^2|$, and three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$. These oscillation parameters are known to be within

$$\begin{aligned} \Delta m_{\odot}^2 &= (7.00 - 8.09) \times 10^{-5} \text{ eV}^2, \\ \Delta m_{\text{A}}^2 &= (2.276 - 2.695) \times 10^{-3} \text{ eV}^2, \\ \sin^2 \theta_{12} &= (0.267 - 0.344), \\ \sin^2 \theta_{23} &= (0.342 - 0.667), \\ \sin^2 \theta_{13} &= (0.0156 - 0.0299), \end{aligned} \quad (6)$$

at 3σ [12], respectively. We therefore have the relations $\Delta m_{\odot}^2 \ll \Delta m_{\text{A}}^2$ and $|U_{e1}|^2 > |U_{e2}|^2 \gg |U_{e3}|^2$.

The effective Majorana mass $|m_{ee}|$ can be written as

$$|m_{ee}| = \left| m_{ee}^{(1)} + |m_{ee}^{(2)}|e^{2i\alpha} + |m_{ee}^{(3)}|e^{2i\beta} \right|, \quad (7)$$

where

$$\begin{aligned} |m_{ee}^{(1)}| &\equiv m_1 |U_{e1}|^2 = m_1 c_{12}^2 c_{13}^2, \\ |m_{ee}^{(2)}| &\equiv m_2 |U_{e2}|^2 = m_2 s_{12}^2 c_{13}^2, \\ |m_{ee}^{(3)}| &\equiv m_3 |U_{e3}|^2 = m_3 s_{13}^2, \end{aligned} \quad (8)$$

and

$$|U_{e1}|^2 + |U_{e2}|^2 + |U_{e3}|^2 = 1. \quad (9)$$

When obtaining the isocontours of the effective Majorana mass $|m_{ee}|$ in the $\beta\alpha$ -plane, the order of three $|m_{ee}^{(i)}|$ according to size becomes an important issue, as will be discussed in detail in section 3 and 4. The effective Majorana mass $|m_{ee}|$ is invariant under $(\beta, \alpha) \rightarrow (-\beta, -\alpha)$ as well as under $\beta \rightarrow \beta + n\pi, (n \in \mathbf{Z})$ or $\alpha \rightarrow \alpha + m\pi, (m \in \mathbf{Z})$. The $|m_{ee}|$ depends on seven parameters: mixing angles $(\theta_{12}, \theta_{13})$, neutrino mass (m_1, m_2, m_3) , and CP-violating Majorana phases (β, α) . For these seven parameters, $\theta_{12}, \theta_{13}, \Delta m_{\odot}^2$, and Δm_{A}^2 are measured by experiments [12], while the Majorana phases β, α (if Majorana particles) and the absolute neutrino mass scale are not measured.

Recently, an upper limit for the sum of the three light neutrino mass has been reported by Planck measurements of the cosmic microwave background [24, 25],

$$m_1 + m_2 + m_3 < 0.23 \text{ eV} \quad (\text{Planck} + \text{WP} + \text{highL} + \text{BAO}). \quad (10)$$

In this paper, we regard the four parameters, $\theta_{12}, \theta_{13}, \Delta m_\odot^2$, and Δm_A^2 as given quantities. The absolute neutrino mass scale is treated as a free parameter. Then, the effective Majorana mass $|m_{ee}|$ can be considered to be a function of two variables β and α when one sets the size of the absolute neutrino mass scale. There remains two possibilities with respect to the order of m_1, m_2 , and m_3 , that is the normal mass ordering ($m_3 > m_2 > m_1$) and inverted mass ordering ($m_2 > m_1 > m_3$). Characteristic features of each mass ordering are discussed in the following.

In the normal mass ordering ($m_3 > m_2 > m_1$), we take the lightest neutrino mass m_1 as the absolute neutrino mass scale, and regard it as a free parameter,

$$\begin{aligned} m_2 &= \sqrt{m_1^2 + \Delta m_\odot^2}, \\ m_3 &= \sqrt{m_1^2 + \Delta m_\text{A}^2}. \end{aligned} \quad (11)$$

The $|m_{ee}^{(i)}|$ defined in Eqs.(8) are represented by use of the lightest neutrino mass m_1 ,

$$\begin{aligned} |m_{ee}^{(1)}| &= m_1 |U_{e1}|^2, \\ |m_{ee}^{(2)}| &= \sqrt{m_1^2 + \Delta m_\odot^2} |U_{e2}|^2, \\ |m_{ee}^{(3)}| &= \sqrt{m_1^2 + \Delta m_\text{A}^2} |U_{e3}|^2. \end{aligned} \quad (12)$$

Setting the value of the lightest neutrino mass m_1 , we can regard the effective Majorana mass $|m_{ee}|$ as the function of β and α . Now let us consider the order of $|m_{ee}^{(1)}|, |m_{ee}^{(2)}|$, and $|m_{ee}^{(3)}|$ according to size in the normal mass ordering case. For an arbitrary value of the lightest neutrino mass m_1 , the following inequality holds,

$$\frac{|m_{ee}^{(3)}|}{|m_{ee}^{(2)}|} = \frac{|U_{e3}|^2 m_3}{|U_{e2}|^2 m_2} \leq \frac{|U_{e3}|^2}{|U_{e2}|^2} \sqrt{\frac{\Delta m_\text{A}^2}{\Delta m_\odot^2}}. \quad (13)$$

By use of the experimental values [12], Eq.(6), one has the relation $|m_{ee}^{(3)}| < |m_{ee}^{(2)}|$ at 3σ . Taking account of this, we assume in this paper that the inequality,

$$|m_{ee}^{(3)}| < |m_{ee}^{(2)}|, \quad (14)$$

is satisfied for an arbitrary value of m_1 . We examine the order of $|m_{ee}^{(1)}|, |m_{ee}^{(2)}|$, and $|m_{ee}^{(3)}|$ according to size when the value of the lightest neutrino mass m_1 is varied. When taking the $m_1 \rightarrow 0$ limit, we have

$$|m_{ee}^{(1)}| < |m_{ee}^{(3)}|. \quad (15)$$

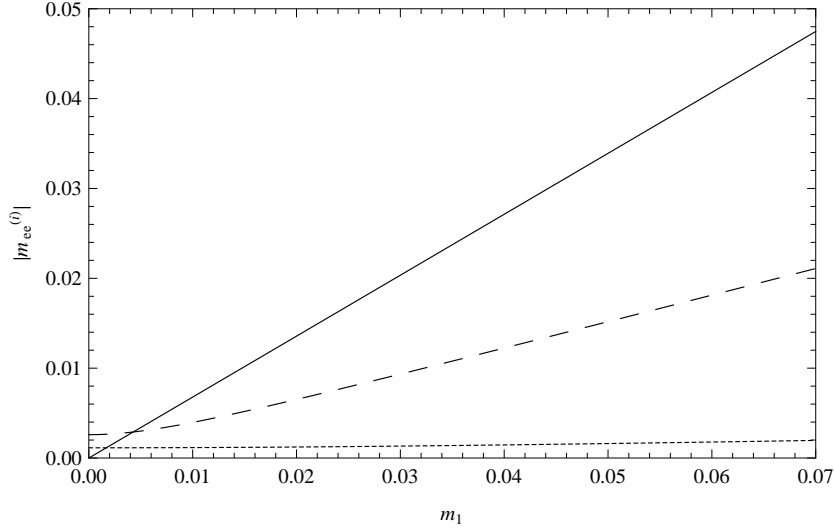


Figure 1: The value (in eV) of $|m_{ee}^{(1)}|$ (Solid), $|m_{ee}^{(2)}|$ (long-dashed), and $|m_{ee}^{(3)}|$ (dotted) as a function of the lightest neutrino mass m_1 (in eV), respectively, for the normal mass ordering case.

On the other hand, for $m_1^2 \gg \Delta m_\odot^2$, it follows

$$|m_{ee}^{(2)}| < |m_{ee}^{(1)}|, \quad (16)$$

because of $|U_{e2}|^2 < |U_{e1}|^2$ and $m_2 = m_1 \sqrt{1 + \Delta m_\odot^2/m_1^2} \approx m_1$. Therefore, the $|m_{ee}^{(1)}|$ which is proportional to m_1 changes from zero to the maximum of $|m_{ee}^{(i)}|$ ($i = 1, 2, 3$) when the value of the lightest neutrino mass m_1 is varied. For a concrete illustration, we show in Fig.1 the relation of the lightest neutrino mass m_1 and $|m_{ee}^{(i)}|$ ($i = 1, 2, 3$) when the values of parameters are set as $\Delta m_\odot^2 = 7.55 \times 10^{-5} \text{eV}^2$, $\Delta m_A^2 = 2.486 \times 10^{-3} \text{eV}^2$, $\sin^2 \theta_{12} = 0.306$, and $\sin^2 \theta_{13} = 0.0228$. After section 3, we do not specify the values of Δm_\odot^2 , Δm_A^2 , $\sin^2 \theta_{12}$, and $\sin^2 \theta_{13}$. For the lightest neutrino mass m_1 which is regarded as a free parameter, we do not give the value of m_1 but specify the size of $|m_{ee}^{(1)}|$ which is proportional to m_1 .

In the inverted mass ordering ($m_2 > m_1 > m_3$), we take the lightest neutrino mass m_3 as the absolute neutrino mass scale, and regard it as a free parameter,

$$\begin{aligned} m_2 &= \sqrt{m_3^2 + \Delta m_\odot^2 + \Delta m_A^2}, \\ m_1 &= \sqrt{m_3^2 + \Delta m_A^2}. \end{aligned} \quad (17)$$

The $|m_{ee}^{(i)}|$ defined in Eq.(8) are represented by use of the lightest neutrino mass m_3 ,

$$\begin{aligned} |m_{ee}^{(1)}| &= \sqrt{m_3^2 + \Delta m_A^2} |U_{e1}|^2, \\ |m_{ee}^{(2)}| &= \sqrt{m_3^2 + \Delta m_\odot^2 + \Delta m_A^2} |U_{e2}|^2, \\ |m_{ee}^{(3)}| &= m_3 |U_{e3}|^2. \end{aligned} \quad (18)$$

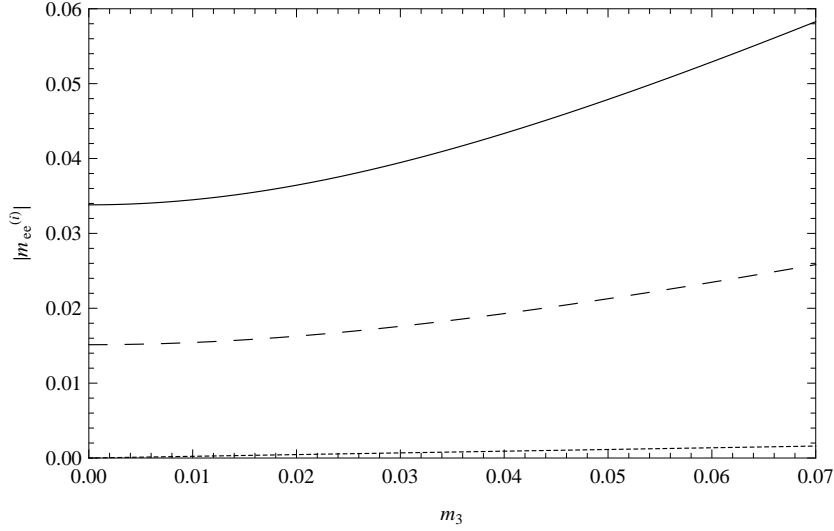


Figure 2: The value (in eV) of $|m_{ee}^{(1)}|$ (Solid), $|m_{ee}^{(2)}|$ (long-dashed), and $|m_{ee}^{(3)}|$ (dotted) as a function of the lightest neutrino mass m_3 (in eV), respectively, for the inverted mass ordering case.

Setting the value of the lightest neutrino mass m_3 , we can regard the effective Majorana mass $|m_{ee}|$ as a function of β and α . For an arbitrary value of the lightest neutrino mass m_3 , we have

$$|m_{ee}^{(1)}| > |m_{ee}^{(2)}| \gg |m_{ee}^{(3)}|, \quad (19)$$

in the inverted mass ordering because of $|U_{e1}|^2 > |U_{e2}|^2 \gg |U_{e3}|^2$ and $m_1 \approx m_2$. The later relation, $m_1 \approx m_2$, comes from the following inequality,

$$1 < \frac{m_2}{m_1} = \sqrt{1 + \frac{\Delta m_{\odot}^2}{m_3^2 + \Delta m_A^2}} < \sqrt{1 + \frac{\Delta m_{\odot}^2}{\Delta m_A^2}} \approx 1.018. \quad (20)$$

For a concrete illustration, we show in Fig.2 the relation of the lightest neutrino mass m_3 and $|m_{ee}^{(i)}|$ ($i = 1, 2, 3$) when the values of Δm_{\odot}^2 , Δm_A^2 , $\sin^2 \theta_{12}$, and $\sin^2 \theta_{13}$ are the same with those of Fig.1. As discussed before, we do not specify the values of Δm_{\odot}^2 , Δm_A^2 , $\sin^2 \theta_{12}$, and $\sin^2 \theta_{13}$ after section 3.

3 Majorana phases for the normal mass ordering

One can expect that the $0\nu\beta\beta$ experiment will bring the information on the Majorane phases β and α in the future. In this chapter, we derive the equation representing the isocontour of the effective Majorana mass $|m_{ee}|$ in the $\beta\alpha$ -plane in the normal mass ordering case ($m_3 > m_2 > m_1$). As discussed in section 2, the lightest neutrino mass m_1 is treated as a free parameter, and when we set the value of m_1 , the effective Majorana mass $|m_{ee}|$ is considered as a function of β and α . If the value of $|m_{ee}|$ is determined, the

constraint on β and α (isocontour of $|m_{ee}|$ in the $\beta\alpha$ -plane) is obtained; however, one can not obtain both values of β and α simultaneously. We first obtain the isocontour of $|m_{ee}|$ around the point of maximum $|m_{ee}|$ in the $\beta\alpha$ -plane, and next those around the point of minimum $|m_{ee}|$. The point of maximum or minimum $|m_{ee}|$ will be found in the region,

$$-\frac{\pi}{2} < \beta \leq \frac{\pi}{2}, \quad -\frac{\pi}{2} \leq \alpha < \frac{\pi}{2}. \quad (21)$$

In the last subsection, we look for the isocontour except around the point of maximum or minimum $|m_{ee}|$. All these calculations are done on the assumption that the relation $|m_{ee}^{(3)}| < |m_{ee}^{(2)}|$ holds as discussed in section 2.

3.1 Around maximum value of $|m_{ee}|$.

For an arbitrary value of the lightest neutrino mass m_1 , the effective Majorana mass $|m_{ee}|$ takes a maximum value,

$$|m_{ee}|_{\max} = |m_{ee}^{(1)}| + |m_{ee}^{(2)}| + |m_{ee}^{(3)}|, \quad (22)$$

at the point $(\beta, \alpha) = (0, 0)$ in the $\beta\alpha$ -plane. In order to obtain the isocontour around $(\beta, \alpha) = (0, 0)$, we expand $|m_{ee}|^2$ around this point. Since the dependence of $|m_{ee}|^2$ on β and α is

$$\begin{aligned} |m_{ee}|^2 = & |m_{ee}^{(1)}|^2 + |m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2 + 2|m_{ee}^{(1)}||m_{ee}^{(2)}|\cos 2\alpha + 2|m_{ee}^{(1)}||m_{ee}^{(3)}|\cos 2\beta \\ & + 2|m_{ee}^{(2)}||m_{ee}^{(3)}|\cos 2\alpha \cos 2\beta + 2|m_{ee}^{(2)}||m_{ee}^{(3)}|\sin 2\alpha \sin 2\beta, \end{aligned} \quad (23)$$

there appears no odd order term on the two variables β and α in that expansion. If one includes the terms up to second order on the two variables β and α in the expansion and disregards the higher orders, it becomes

$$\begin{aligned} |m_{ee}|^2 \approx & |m_{ee}|_{\max}^2 - 4\beta^2 \left\{ (|m_{ee}^{(1)}| + |m_{ee}^{(2)}|)|m_{ee}^{(3)}| \right\} \\ & - 4\alpha^2 \left\{ (|m_{ee}^{(1)}| + |m_{ee}^{(3)}|)|m_{ee}^{(2)}| \right\} + 8\beta\alpha |m_{ee}^{(2)}||m_{ee}^{(3)}|, \end{aligned} \quad (24)$$

or

$$\begin{pmatrix} \beta & \alpha \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \approx J, \quad (25)$$

where

$$\begin{aligned} a &\equiv (|m_{ee}^{(1)}| + |m_{ee}^{(2)}|)|m_{ee}^{(3)}| > 0, \\ b &\equiv -|m_{ee}^{(2)}||m_{ee}^{(3)}| < 0, \\ c &\equiv (|m_{ee}^{(1)}| + |m_{ee}^{(3)}|)|m_{ee}^{(2)}| > 0, \\ J &\equiv \frac{1}{4} \left\{ |m_{ee}|_{\max}^2 - |m_{ee}|^2 \right\} > 0. \end{aligned} \quad (26)$$

A real symmetric matrix can be diagonalized by a rotational matrix,

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix}, \quad (27)$$

where

$$\begin{aligned} \tan \theta &= \frac{c - a - \sqrt{(a - c)^2 + 4b^2}}{2b}, & (b \neq 0), \\ \tan 2\theta &= \frac{2b}{a - c}, & (a - c \neq 0), \\ \lambda_{\pm} &= \frac{(a + c) \pm \sqrt{(a - c)^2 + 4b^2}}{2}. \end{aligned} \quad (28)$$

If $ac - b^2 > 0$ and $a + c > 0$ are satisfied, one has $\lambda_+ > \lambda_- > 0$.

Since $a + c > 0$ and $ac - b^2 > 0$ are satisfied from Eq.(26), a homogeneous polynomial of second order, Eq.(25), represents the ellipse in the $\beta\alpha$ -plane. By diagonalization of the real symmetric matrix, Eq.(25) can be rewritten as

$$\begin{pmatrix} \beta'' & \alpha'' \end{pmatrix} \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} \approx J, \quad (29)$$

where

$$\begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}, \quad (30)$$

and

$$\begin{aligned} \tan \theta &= \frac{-1}{2|m_{ee}^{(2)}||m_{ee}^{(3)}|} \left[|m_{ee}^{(1)}|(|m_{ee}^{(2)}| - |m_{ee}^{(3)}|) \right. \\ &\quad \left. - \sqrt{|m_{ee}^{(1)}|^2(|m_{ee}^{(3)}| - |m_{ee}^{(2)}|)^2 + 4(|m_{ee}^{(2)}||m_{ee}^{(3)}|)^2} \right] > 0, \\ \tan 2\theta &= \frac{2|m_{ee}^{(2)}||m_{ee}^{(3)}|}{|m_{ee}^{(1)}|(|m_{ee}^{(2)}| - |m_{ee}^{(3)}|)} > 0, \\ \lambda_{\pm} &= \frac{1}{2} \left[\left\{ |m_{ee}^{(1)}||m_{ee}^{(3)}| + 2|m_{ee}^{(2)}||m_{ee}^{(3)}| + |m_{ee}^{(1)}||m_{ee}^{(2)}| \right\} \right. \\ &\quad \left. \pm \sqrt{|m_{ee}^{(1)}|^2(|m_{ee}^{(3)}| - |m_{ee}^{(2)}|)^2 + 4(|m_{ee}^{(2)}||m_{ee}^{(3)}|)^2} \right]. \end{aligned} \quad (31)$$

As $\tan \theta$ and $\tan 2\theta (= 2 \tan \theta / (1 - \tan^2 \theta))$ are both positive, we have $0 < \theta < \pi/4$. In the $\beta\alpha$ -plane, the isocontour around the point of maximum $|m_{ee}|_{\max}$ becomes the ellipse represented by the following normal equation,

$$\frac{\beta'^2}{\left(\sqrt{\frac{J}{\lambda_-}}\right)^2} + \frac{\alpha'^2}{\left(\sqrt{\frac{J}{\lambda_+}}\right)^2} = 1, \quad (32)$$

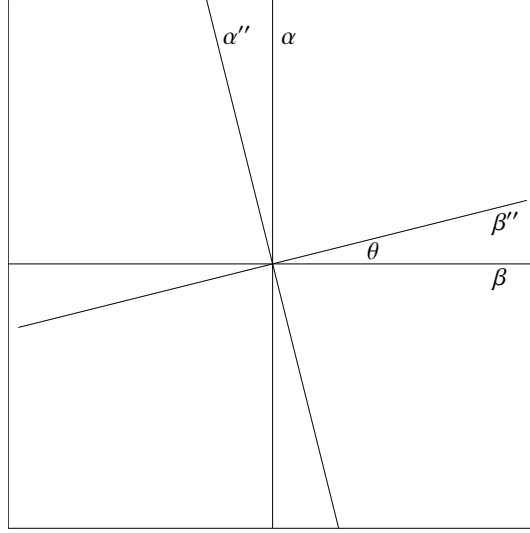


Figure 3: Relation between the β'' axis (the α'' axis) and the β axis (the α axis).

where we have neglected the fourth order of two variables β and α or higher orders. The center of this ellipse is $(\beta, \alpha) = (0, 0)$, and the major axis is $2\sqrt{J/\lambda_-}$, the minor axis $2\sqrt{J/\lambda_+}$. The ratio of the minor axis to the major axis, $\sqrt{\lambda_-/\lambda_+}$, depends on the lightest neutrino mass m_1 , but not on $|m_{ee}|$. The direction of the β'' axis is produced by a counterclockwise rotation of the β axis by the angle $\theta(> 0)$ (see Fig.3). The effective Majorana mass, Eq.(3), is invariant under $\beta \rightarrow \beta + n\pi$, $\alpha \rightarrow \alpha + m\pi$, ($n, m \in \mathbf{Z}$), and consequently the ellipse with the center $(n\pi, m\pi)$ is distributed in the $\beta\alpha$ -plane. When the size of the major axis is a quarter of π , $2\sqrt{J/\lambda_-} = \pi/4$, the effective Majorana mass becomes

$$|m_{ee}| = \sqrt{|m_{ee}|_{\max}^2 - \frac{\pi^2}{16}\lambda_-}. \quad (33)$$

The ratio of the minor axis to the major axis $\sqrt{\lambda_-/\lambda_+}$ reflects the form of the ellipse, the major axis reflects the size of the ellipse, and the angle θ determines the direction of the major axis of the ellipse. Now, we investigate how $\sqrt{\lambda_-/\lambda_+}$ and θ change when the value of the lightest neutrino mass m_1 is varied. We also examine the relation between the effective Majorana mass $|m_{ee}|$ and the major axis of the ellipse which is the isocontour of $|m_{ee}|$. The expressions of λ_{\pm} or $\tan\theta$ are, however, too complicated to study these problems practically. As discussed in section 2, the $|m_{ee}^{(1)}|$ changes from zero to the maximum of $|m_{ee}^{(i)}|$ ($i = 1, 2, 3$) when the value of the lightest neutrino mass m_1 is varied. When the value of m_1 (or $|m_{ee}^{(1)}| = m_1|U_{e1}|^2$) is in some regions, λ_{\pm} or $\tan\theta$ can be expressed simply, and we give three instances in the following.

The first case is that the value of the lightest neutrino mass m_1 is large enough to satisfy inequality,

$$|m_{ee}^{(1)}| > |m_{ee}^{(2)}| \gg |m_{ee}^{(3)}|, \quad \text{and} \quad m_1 \approx m_2. \quad (34)$$

Owing to this inequality, the terms of order $O(|m_{ee}^{(3)}|^2)$ can be neglected,

$$\begin{aligned}\tan \theta &\approx \frac{|m_{ee}^{(3)}|}{|m_{ee}^{(1)}|} = \frac{m_3}{m_1} \frac{|U_{e3}|^2}{|U_{e1}|^2} \ll 1, \\ \lambda_+ &\approx |m_{ee}^{(1)}||m_{ee}^{(2)}| + |m_{ee}^{(2)}||m_{ee}^{(3)}|, \\ \lambda_- &\approx (|m_{ee}^{(1)}| + |m_{ee}^{(2)}|)|m_{ee}^{(3)}|. \end{aligned} \quad (35)$$

The ratio of the minor axis to the major axis can be written as

$$\sqrt{\frac{\lambda_-}{\lambda_+}} \approx \sqrt{\frac{(|m_{ee}^{(1)}| + |m_{ee}^{(2)}|)|m_{ee}^{(3)}|}{|m_{ee}^{(1)}||m_{ee}^{(2)}|}} \approx \sqrt{\frac{m_3}{m_1}} \sqrt{\frac{|U_{e3}|^2(1 - |U_{e3}|^2)}{|U_{e1}|^2|U_{e2}|^2}}, \quad (36)$$

where we have used $m_1 \approx m_2$. In the region where the condition Eq.(34) is satisfied, both the angle θ and the ratio $\sqrt{\lambda_-/\lambda_+}$ increase when the value of the the lightest neutrino mass m_1 is decreased. When the size of the major axis of the ellipse is $\pi/4$, the effective Majorana mass Eq.(33) becomes

$$|m_{ee}| \approx |m_{ee}|_{\max} - \frac{\pi^2}{32} |m_{ee}^{(3)}| = |m_{ee}^{(1)}| + |m_{ee}^{(2)}| + \left(1 - \frac{\pi^2}{32}\right) |m_{ee}^{(3)}|. \quad (37)$$

In other words, when the effective Majorana mass takes the value Eq.(37), the corresponding isocontour is the ellipse represented by

$$\frac{\beta'^2}{\left(\frac{\pi}{8}\right)^2} + \frac{\alpha'^2}{\left(\frac{\pi}{8} \sqrt{\frac{(|m_{ee}^{(1)}| + |m_{ee}^{(2)}|)|m_{ee}^{(3)}|}{|m_{ee}^{(1)}||m_{ee}^{(2)}|}}\right)^2} = 1. \quad (38)$$

The second case is that the the lightest neutrino mass m_1 has smaller value than the first case and satisfies

$$|m_{ee}^{(1)}| = |m_{ee}^{(2)}| + |m_{ee}^{(3)}|. \quad (39)$$

In this case, we have

$$\begin{aligned}\tan \theta &= \frac{|m_{ee}^{(3)}|}{|m_{ee}^{(2)}|} < 1, \\ \sqrt{\frac{\lambda_-}{\lambda_+}} &= \sqrt{\frac{2|m_{ee}^{(2)}||m_{ee}^{(3)}|}{2|m_{ee}^{(2)}||m_{ee}^{(3)}| + |m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2}} < 1, \end{aligned} \quad (40)$$

and both the angle θ and the ratio $\sqrt{\lambda_-/\lambda_+}$ are larger than the first case, respectively. When the size of the major axis of the ellipse is $\pi/4$, the effective Majorana mass becomes

$$|m_{ee}| \approx |m_{ee}|_{\max} - \frac{\pi^2}{32} \frac{|m_{ee}^{(2)}||m_{ee}^{(3)}|}{(|m_{ee}^{(2)}| + |m_{ee}^{(3)}|)}. \quad (41)$$

The third case is that the lightest neutrino mass m_1 has much smaller value than the second case and satisfies

$$|m_{ee}^{(2)}| > |m_{ee}^{(3)}| \gg |m_{ee}^{(1)}|. \quad (42)$$

In this case, the terms of order $O(|m_{ee}^{(1)}|^2)$ can be neglected and one has

$$\tan \theta \approx 1 - \frac{(|m_{ee}^{(2)}| - |m_{ee}^{(3)}|)}{2|m_{ee}^{(2)}||m_{ee}^{(3)}|}|m_{ee}^{(1)}|, \quad (43)$$

or

$$\theta \approx \frac{\pi}{4} - \frac{(|m_{ee}^{(2)}| - |m_{ee}^{(3)}|)}{4|m_{ee}^{(2)}||m_{ee}^{(3)}|}|m_{ee}^{(1)}| \approx \frac{\pi}{4}, \quad (44)$$

and

$$\sqrt{\frac{\lambda_-}{\lambda_+}} \approx \sqrt{\frac{(|m_{ee}^{(2)}| + |m_{ee}^{(3)}|)}{4|m_{ee}^{(2)}||m_{ee}^{(3)}|}|m_{ee}^{(1)}|}. \quad (45)$$

The angle θ is larger than the second case, while the ratio $\sqrt{\lambda_-/\lambda_+}$ is much smaller than the second case. In the region where the condition Eq.(42) is satisfied, the angle θ increases and the ratio $\sqrt{\lambda_-/\lambda_+}$ decreases when the value of the the lightest neutrino mass m_1 is decreased. In the $m_1 \rightarrow 0$ limit, we have

$$\theta \rightarrow \frac{\pi}{4}, \quad \sqrt{\frac{\lambda_-}{\lambda_+}} \rightarrow 0, \quad (m_1 \rightarrow 0). \quad (46)$$

When the size of the major axis of the ellipse is $\pi/4$, the effective Majorana mass becomes

$$|m_{ee}| \approx |m_{ee}|_{\max} - \frac{\pi^2}{64}|m_{ee}^{(1)}|. \quad (47)$$

3.2 Around minimum value of $|m_{ee}|$.

The position of the point (β, α) giving the minimum value of $|m_{ee}|$ shifts according to the value of the lightest neutrino mass m_1 (or $|m_{ee}^{(1)}|$), because $|m_{ee}^{(1)}|$ changes from zero to the maximum of $|m_{ee}^{(i)}|$ ($i = 1, 2, 3$) when the value of m_1 is varied. In studying the minimum value of $|m_{ee}|$, it is convenient to divide $|m_{ee}^{(1)}|$ into the following three regions,

- region (A); $|m_{ee}^{(2)}| + |m_{ee}^{(3)}| < |m_{ee}^{(1)}|,$
- region (B); $|m_{ee}^{(2)}| - |m_{ee}^{(3)}| < |m_{ee}^{(1)}| < |m_{ee}^{(2)}| + |m_{ee}^{(3)}|,$
- region (C); $|m_{ee}^{(1)}| < |m_{ee}^{(2)}| - |m_{ee}^{(3)}|.$

The minimum value of $|m_{ee}|$ in each of the three regions becomes [21]

- region (A); $|m_{ee}|_{\min} = |m_{ee}^{(1)}| - |m_{ee}^{(2)}| - |m_{ee}^{(3)}|,$
- region (B); $|m_{ee}|_{\min} = 0,$
- region (C); $|m_{ee}|_{\min} = |m_{ee}^{(2)}| - |m_{ee}^{(3)}| - |m_{ee}^{(1)}|.$

In this subsection, we obtain the isocontour of $|m_{ee}|$ in the $\beta\alpha$ -plane around the point of the minimum $|m_{ee}|$ in each of the three regions (A), (B), and (C). Moreover, the characteristic features of the isocontour on the boundary between the region (A) and (B), or between (B) and (C), will be discussed.

3.2.1 The region (A) : $|m_{ee}^{(2)}| + |m_{ee}^{(3)}| < |m_{ee}^{(1)}|.$

In the region (A), the effective Majorana mass has a minimum value, $|m_{ee}|_{\min},$

$$|m_{ee}|_{\min} = |m_{ee}^{(1)}| - |m_{ee}^{(2)}| - |m_{ee}^{(3)}|, \quad (48)$$

at the point $(\beta, \alpha) = (\pi/2, -\pi/2).$ We expand $|m_{ee}|^2$ around this point to obtain the isocontour around the point. In a similar way to the discussion in section 3.1, there appears no odd order term on the two variables $(\beta - \pi/2)$ and $(\alpha + \pi/2)$ in this expansion around $(\beta, \alpha) = (\pi/2, -\pi/2).$ Up to the second order, the expansion results

$$\begin{aligned} |m_{ee}|^2 \approx & |m_{ee}|_{\min}^2 + 4(\beta - \frac{\pi}{2})^2 \left\{ (|m_{ee}^{(1)}| - |m_{ee}^{(2)}|) |m_{ee}^{(3)}| \right\} \\ & + 4(\alpha + \frac{\pi}{2})^2 \left\{ (|m_{ee}^{(1)}| - |m_{ee}^{(3)}|) |m_{ee}^{(2)}| \right\} \\ & + 8(\beta - \frac{\pi}{2})(\alpha + \frac{\pi}{2}) |m_{ee}^{(2)}| |m_{ee}^{(3)}|, \end{aligned} \quad (49)$$

or

$$\begin{pmatrix} \beta' & \alpha' \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} \approx K, \quad (50)$$

where

$$\begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} \beta - \frac{\pi}{2} \\ \alpha + \frac{\pi}{2} \end{pmatrix}, \quad (51)$$

$$\begin{aligned} a &= (|m_{ee}^{(1)}| - |m_{ee}^{(2)}|) |m_{ee}^{(3)}| > 0, \\ b &= |m_{ee}^{(2)}| |m_{ee}^{(3)}| > 0, \\ c &= (|m_{ee}^{(1)}| - |m_{ee}^{(3)}|) |m_{ee}^{(2)}| > 0, \\ K &= \frac{1}{4} \left\{ |m_{ee}|^2 - |m_{ee}|_{\min}^2 \right\} > 0, \end{aligned} \quad (52)$$

and one has $ac - b^2 > 0$. By diagonalizing the real symmetric matrix as discussed in section 3.1, we can rewrite Eq.(50) as

$$\begin{pmatrix} \beta'' & \alpha'' \end{pmatrix} \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} \approx K, \quad (53)$$

where

$$\begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} (\beta - \frac{\pi}{2}) \\ (\alpha + \frac{\pi}{2}) \end{pmatrix}, \quad (54)$$

and

$$\begin{aligned} \tan \theta &= \frac{+1}{2|m_{ee}^{(2)}||m_{ee}^{(3)}|} \left[|m_{ee}^{(1)}|(|m_{ee}^{(2)}| - |m_{ee}^{(3)}|) \right. \\ &\quad \left. - \sqrt{|m_{ee}^{(1)}|^2(|m_{ee}^{(3)}| - |m_{ee}^{(2)}|)^2 + 4(|m_{ee}^{(2)}||m_{ee}^{(3)}|)^2} \right] < 0, \\ \tan 2\theta &= -\frac{2|m_{ee}^{(2)}||m_{ee}^{(3)}|}{|m_{ee}^{(1)}|(|m_{ee}^{(2)}| - |m_{ee}^{(3)}|)} < 0, \\ \lambda_{\pm} &= \frac{1}{2} \left[\left\{ |m_{ee}^{(1)}||m_{ee}^{(3)}| - 2|m_{ee}^{(2)}||m_{ee}^{(3)}| + |m_{ee}^{(1)}||m_{ee}^{(2)}| \right\} \right. \\ &\quad \left. \pm \sqrt{|m_{ee}^{(1)}|^2(|m_{ee}^{(3)}| - |m_{ee}^{(2)}|)^2 + 4(|m_{ee}^{(2)}||m_{ee}^{(3)}|)^2} \right]. \end{aligned} \quad (55)$$

Because of $\tan \theta < 0$ and $\tan 2\theta < 0$, one has $-\pi/4 < \theta < 0$. In the region (A), $|m_{ee}^{(2)}| + |m_{ee}^{(3)}| < |m_{ee}^{(1)}|$, the isocontour around the point of the minimum $|m_{ee}|_{\min} = |m_{ee}^{(1)}| - |m_{ee}^{(2)}| - |m_{ee}^{(3)}|$ becomes the ellipse represented by the following equation,

$$\frac{\beta'^2}{\left(\sqrt{\frac{K}{\lambda_-}}\right)^2} + \frac{\alpha'^2}{\left(\sqrt{\frac{K}{\lambda_+}}\right)^2} = 1, \quad (56)$$

where the fourth order of two variables $(\beta - \pi/2)$ and $(\alpha + \pi/2)$ or higher orders have been neglected. This ellipse has the center at $(\beta, \alpha) = (\pi/2, -\pi/2)$, and the major axis, $2\sqrt{K/\lambda_-}$, the minor axis, $2\sqrt{K/\lambda_+}$. The direction of the β'' axis is produced by a clockwise rotation of the β axis by the angle $|\theta|(> 0)$. From the invariance of $|m_{ee}|$, Eq.(3), the ellipse with the center $(\pi/2 + n\pi, -\pi/2 + m\pi)$ is distributed in the $\beta\alpha$ -plane. When the size of the major axis is a quarter of π , $2\sqrt{K/\lambda_-} = \pi/4$, the effective Majorana mass becomes

$$|m_{ee}| = \sqrt{|m_{ee}|_{\min}^2 + \frac{\pi^2}{16}\lambda_-}. \quad (57)$$

As considered in section 3.1, we investigate how the ratio of the minor axis to the major axis, $\sqrt{\lambda_-/\lambda_+}$, reflecting the form of the ellipse, and the angle θ determining the direction of the major axis of the ellipse change when the value of the lightest

neutrino mass m_1 is varied. Although the expressions of λ_{\pm} or $\tan \theta$ in the region (A) are very complicated, these expressions are simplified when the value of m_1 (or $|m_{ee}^{(1)}| = m_1|U_{e1}|^2$) satisfies the following,

$$|m_{ee}^{(1)}| > |m_{ee}^{(2)}| \gg |m_{ee}^{(3)}|, \quad m_1 \approx m_2. \quad (58)$$

Neglecting the order $O(|m_{ee}^{(3)}|^2)$, we have

$$\begin{aligned} \theta &\approx -\frac{|m_{ee}^{(3)}|}{|m_{ee}^{(1)}|} = -\frac{m_3|U_{e3}|^2}{m_1|U_{e1}|^2} < 0, \quad |\theta| \ll 1, \\ \sqrt{\frac{\lambda_-}{\lambda_+}} &\approx \sqrt{\frac{(|m_{ee}^{(1)}| - |m_{ee}^{(2)}|)|m_{ee}^{(3)}|}{|m_{ee}^{(1)}||m_{ee}^{(2)}|}} \approx \sqrt{\frac{m_3}{m_1}} \sqrt{\frac{|U_{e3}|^2(|U_{e1}|^2 - |U_{e2}|^2)}{|U_{e1}|^2|U_{e2}|^2}}. \end{aligned} \quad (59)$$

In the region where the condition Eq.(58) is satisfied, both the absolute value of the angle $|\theta|$ and the ratio $\sqrt{\lambda_-/\lambda_+}$ increases when the value of the lightest neutrino mass m_1 is decreased. When the size of the major axis of the ellipse is $\pi/4$, the effective Majorana mass Eq.(57) becomes

$$|m_{ee}| \approx |m_{ee}|_{\min} + \frac{\pi^2}{32}|m_{ee}^{(3)}| = |m_{ee}^{(1)}| - |m_{ee}^{(2)}| - \left(1 - \frac{\pi^2}{32}\right)|m_{ee}^{(3)}|. \quad (60)$$

3.2.2 On the boundary between the region (A) and region (B).

Let us focus on the boundary between the region (A) and region (B),

$$|m_{ee}^{(1)}| = |m_{ee}^{(2)}| + |m_{ee}^{(3)}|. \quad (61)$$

On this boundary, the effective Majorana mass has a minimum value $|m_{ee}|_{\min} = 0$ at $(\beta, \alpha) = (\pi/2, -\pi/2)$. When the $|m_{ee}|^2$ is expanded around the point $(\beta, \alpha) = (\pi/2, -\pi/2)$, there appears no odd order term on the two variables $(\beta - \pi/2)$ and $(\alpha + \pi/2)$ in the expansion. The expansion up to the second order takes the same form with Eq.(49). Because of $c - a > 0$ and $ac - b^2 = 0$ on this boundary, one has

$$\tan \theta = -\frac{|m_{ee}^{(3)}|}{|m_{ee}^{(2)}|} < 0, \quad \lambda_+ = |m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2, \quad \lambda_- = 0, \quad (62)$$

and $-\pi/4 < \theta < 0$. This fact shows that, within the approximation up to the second order, the isocontour around the point of $|m_{ee}|_{\min}$ is not an ellipse but a straight line,

$$0 + \frac{\alpha'^2}{\left(\sqrt{\frac{K}{\lambda_+}}\right)^2} = 1. \quad (63)$$

If one takes into account the higher order terms such as $O((\beta - \pi/2)^4)$, $O((\beta - \pi/2)^3(\alpha + \pi/2))$, $O((\beta - \pi/2)^2(\alpha + \pi/2)^2)$, and $O((\alpha + \pi/2)^4)$, the isocontour would become a closed line.

3.2.3 The region (B) : $|m_{ee}^{(2)}| - |m_{ee}^{(3)}| < |m_{ee}^{(1)}| < |m_{ee}^{(2)}| + |m_{ee}^{(3)}|$.

In the region (B), the effective Majorana mass has a minimum value $|m_{ee}|_{\min}$,

$$|m_{ee}|_{\min} = 0. \quad (64)$$

The reason why the minimum value becomes zero [16] is explained as follows [21, 18]. The effective Majorana mass $|m_{ee}|$ is given by the absolute value of the sum of three complex vectors, $|m_{ee}^{(1)}|$, $|m_{ee}^{(2)}|e^{2i\alpha}$, and $|m_{ee}^{(3)}|e^{2i\beta}$. If these three complex vectors form a triangle, $|m_{ee}^{(1)}| + |m_{ee}^{(2)}|e^{2i\alpha} + |m_{ee}^{(3)}|e^{2i\beta} = 0$, one has $|m_{ee}| = 0$ [21]. Now, we look for the necessary condition to form a triangle using the assumption $|m_{ee}^{(3)}| < |m_{ee}^{(2)}|$ made in section 2. In the case of $|m_{ee}^{(2)}| < |m_{ee}^{(1)}|$, the necessary condition is $|m_{ee}^{(1)}| < |m_{ee}^{(2)}| + |m_{ee}^{(3)}|$, while in the case of $|m_{ee}^{(1)}| < |m_{ee}^{(2)}|$, the necessary condition is $|m_{ee}^{(2)}| < |m_{ee}^{(1)}| + |m_{ee}^{(3)}|$. Therefore, the necessary condition to form a triangle is $|m_{ee}^{(2)}| - |m_{ee}^{(3)}| < |m_{ee}^{(1)}| < |m_{ee}^{(2)}| + |m_{ee}^{(3)}|$, and we can show that the effective Majorana mass has a minimum value zero in the region (B).

Next, we seek the values of Majorana phases β_c and α_c which satisfy $|m_{ee}| = 0$ [14]. Since $|m_{ee}|$ is invariant under $(\beta, \alpha) \rightarrow (-\beta, -\alpha)$,

$$|m_{ee}|(\beta, \alpha) = |m_{ee}|(-\beta, -\alpha), \quad (65)$$

we restrict β_c and α_c within

$$-\frac{\pi}{2} < \beta_c < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \alpha_c < 0. \quad (66)$$

Solving the equation,

$$\begin{aligned} 0 &= |m_{ee}^{(1)}| + |m_{ee}^{(2)}|e^{2i\alpha_c} + |m_{ee}^{(3)}|e^{2i\beta_c} \\ &= \left\{ |m_{ee}^{(1)}| + |m_{ee}^{(2)}| \cos 2\alpha_c + |m_{ee}^{(3)}| \cos 2\beta_c \right\} + i \left\{ |m_{ee}^{(2)}| \sin 2\alpha_c + |m_{ee}^{(3)}| \sin 2\beta_c \right\} \end{aligned} \quad (67)$$

we obtain

$$\cos 2\alpha_c = \frac{|m_{ee}^{(3)}|^2 - |m_{ee}^{(1)}|^2 - |m_{ee}^{(2)}|^2}{2|m_{ee}^{(1)}||m_{ee}^{(2)}|}, \quad (68)$$

and then

$$\cos 2\beta_c = \frac{|m_{ee}^{(2)}|^2 - |m_{ee}^{(3)}|^2 - |m_{ee}^{(1)}|^2}{2|m_{ee}^{(1)}||m_{ee}^{(3)}|}. \quad (69)$$

The Eq.(68) and the assumption $|m_{ee}^{(3)}| < |m_{ee}^{(2)}|$ lead to $-\pi/2 < \alpha_c < -\pi/4$, and then $\sin 2\alpha_c < 0$. From Eq.(67), the product of $\sin 2\alpha_c$ and $\sin 2\beta_c$ must be negative, $\sin 2\alpha_c \sin 2\beta_c < 0$ [22]. Consequently, $\sin 2\beta_c$ must be positive and $0 < \beta_c < \frac{\pi}{2}$. We finally find that β_c and α_c satisfy Eq.(69) and Eq.(68), respectively, and they are within

$$0 < \beta_c < \frac{\pi}{2}, \quad -\frac{\pi}{2} < \alpha_c < -\frac{\pi}{4}. \quad (70)$$

We expand $|m_{ee}|^2$ around the point (β_c, α_c) and it is approximated by the terms up to second order,

$$|m_{ee}|^2 \approx 4(\beta - \beta_c)^2 |m_{ee}^{(3)}|^2 + 4(\alpha - \alpha_c)^2 |m_{ee}^{(2)}|^2 + 8(\beta - \beta_c)(\alpha - \alpha_c) \left\{ |m_{ee}^{(2)}| |m_{ee}^{(3)}| \cos(2\alpha_c - 2\beta_c) \right\}, \quad (71)$$

or

$$\begin{pmatrix} \beta' & \alpha' \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} \approx K, \quad (72)$$

where

$$\begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} \beta - \beta_c \\ \alpha - \alpha_c \end{pmatrix}, \quad (73)$$

$$\begin{aligned} a &= |m_{ee}^{(3)}|^2 > 0, \\ b &= \frac{1}{2} \left\{ |m_{ee}^{(1)}|^2 - |m_{ee}^{(2)}|^2 - |m_{ee}^{(3)}|^2 \right\}, \\ c &= |m_{ee}^{(2)}|^2 > 0, \\ K &= \frac{1}{4} |m_{ee}|^2, \end{aligned} \quad (74)$$

and then $ac - b^2 > 0$ and $c - a > 0$. As in section 3.1, Eq.(72) can be rewritten as

$$\begin{pmatrix} \beta'' & \alpha'' \end{pmatrix} \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} \approx K, \quad (75)$$

where

$$\begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \beta - \beta_c \\ \alpha - \alpha_c \end{pmatrix}, \quad (76)$$

and

$$\begin{aligned} \tan \theta &= \frac{1}{|m_{ee}^{(1)}|^2 - |m_{ee}^{(2)}|^2 - |m_{ee}^{(3)}|^2} \left[|m_{ee}^{(2)}|^2 - |m_{ee}^{(3)}|^2 \right. \\ &\quad \left. - \sqrt{|m_{ee}^{(1)}|^4 + 2|m_{ee}^{(2)}|^4 + 2|m_{ee}^{(3)}|^4 - 2|m_{ee}^{(1)}|^2(|m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2)} \right], \\ \tan 2\theta &= \frac{|m_{ee}^{(1)}|^2 - |m_{ee}^{(2)}|^2 - |m_{ee}^{(3)}|^2}{|m_{ee}^{(3)}|^2 - |m_{ee}^{(2)}|^2}, \\ \lambda_{\pm} &= \frac{1}{2} \left[|m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2 \right. \\ &\quad \left. \pm \sqrt{|m_{ee}^{(1)}|^4 + 2|m_{ee}^{(2)}|^4 + 2|m_{ee}^{(3)}|^4 - 2|m_{ee}^{(1)}|^2(|m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2)} \right]. \end{aligned} \quad (77)$$

The sign of the angle θ depends on the magnitude of $|m_{ee}^{(1)}|$. When the $|m_{ee}^{(1)}|$ satisfies $\sqrt{|m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2} < |m_{ee}^{(1)}| < |m_{ee}^{(2)}| + |m_{ee}^{(3)}|$, the parameter b becomes positive and

we have $-\pi/4 < \theta < 0$. On the other hand, when the $|m_{ee}^{(1)}|$ satisfies $|m_{ee}^{(2)}| - |m_{ee}^{(3)}| < |m_{ee}^{(1)}| < \sqrt{|m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2}$, the parameter b becomes negative and we have $0 < \theta < \pi/4$. In the region (B), $|m_{ee}^{(2)}| - |m_{ee}^{(3)}| < |m_{ee}^{(1)}| < |m_{ee}^{(2)}| + |m_{ee}^{(3)}|$, the isocontour around the point of the minimum $|m_{ee}|_{\min} = 0$ becomes the ellipse represented by the following equation,

$$\frac{\beta'^2}{\left(\sqrt{\frac{K}{\lambda_-}}\right)^2} + \frac{\alpha'^2}{\left(\sqrt{\frac{K}{\lambda_+}}\right)^2} = 1, \quad (78)$$

where the third order of two variables $(\beta - \beta_c)$ and $(\alpha - \alpha_c)$ or higher orders have been neglected. The center of this ellipse is the point (β_c, α_c) . From the invariance of $|m_{ee}|$, Eq.(3) and Eq.(65), the ellipse with the center $(\pm\beta_c + n\pi, \pm\alpha_c + m\pi)$ is distributed in the $\beta\alpha$ -plane.

The expressions of $\cos 2\alpha_c$, $\cos 2\beta_c$, $\tan \theta$, and λ_{\pm} are very complicated. However, when the value of the lightest neutrino mass m_1 ($|m_{ee}^{(1)}| = m_1|U_{e1}|^2$) satisfies $|m_{ee}^{(1)}|^2 = |m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2$, three complex vectors, $|m_{ee}^{(1)}|$, $|m_{ee}^{(2)}|e^{2i\beta}$, and $|m_{ee}^{(3)}|e^{2i\alpha}$ form a right-angle triangle, and the expressions become very simple,

$$\theta = 0, \quad \lambda_- = a = |m_{ee}^{(3)}|^2, \quad \lambda_+ = c = |m_{ee}^{(2)}|^2, \quad \sqrt{\frac{\lambda_-}{\lambda_+}} = \frac{|m_{ee}^{(3)}|}{|m_{ee}^{(2)}|} < 1, \quad (79)$$

and

$$\cos(2\alpha_c) = -\frac{|m_{ee}^{(2)}|}{|m_{ee}^{(1)}|}, \quad \cos(2\beta_c) = -\frac{|m_{ee}^{(3)}|}{|m_{ee}^{(1)}|}. \quad (80)$$

3.2.4 On the boundary between the region (B) and region (C).

We consider the boundary between the region (B) and region (C),

$$|m_{ee}^{(1)}| = |m_{ee}^{(2)}| - |m_{ee}^{(3)}|. \quad (81)$$

On this boundary, the effective Majorana mass has a minimum value $|m_{ee}|_{\min} = 0$ at $(\beta, \alpha) = (0, -\pi/2)$. When the $|m_{ee}|^2$ is expanded around the point $(\beta, \alpha) = (0, -\pi/2)$, there appears no odd order term on the two variables β and $(\alpha + \pi/2)$ in the expansion. The expansion up to the second order takes the same form to Eq.(71). Because of $c - a > 0$ and $ac - b^2 = 0$ on this boundary, one has

$$\tan \theta = \frac{|m_{ee}^{(3)}|}{|m_{ee}^{(2)}|} < 1, \quad \lambda_+ = |m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2, \quad \lambda_- = 0, \quad (82)$$

and $0 < \theta < \pi/4$. This fact shows that, within the approximation up to the second order, the isocontour around the point of $|m_{ee}|_{\min}$ is not an ellipse but a straight line,

$$0 + \frac{\alpha'^2}{\left(\sqrt{\frac{K}{\lambda_+}}\right)^2} = 1. \quad (83)$$

If one takes into account the higher order terms such as $O(\beta^4)$, $O(\beta^3(\alpha + \pi/2))$, $O(\beta^2(\alpha + \pi/2)^2)$, and $O((\alpha + \pi/2)^4)$, the isocontour would become a closed line.

3.2.5 The region (C) : $|m_{ee}^{(1)}| < |m_{ee}^{(2)}| - |m_{ee}^{(3)}|$

In the region (C), the effective Majorana mass has a minimum value $|m_{ee}|_{\min}$,

$$|m_{ee}|_{\min} = ||m_{ee}^{(1)}| - |m_{ee}^{(2)}| + |m_{ee}^{(3)}||, \quad (84)$$

at the point $(\beta, \alpha) = (0, -\pi/2)$. We expand $|m_{ee}|^2$ around this point and it is approximated by the terms up to second order,

$$\begin{aligned} |m_{ee}|^2 \approx & |m_{ee}|_{\min}^2 + 4\beta^2 \left\{ -(|m_{ee}^{(1)}| - |m_{ee}^{(2)}|)|m_{ee}^{(3)}| \right\} \\ & + 4\left(\alpha + \frac{\pi}{2}\right)^2 \left\{ (|m_{ee}^{(1)}| + |m_{ee}^{(3)}|)|m_{ee}^{(2)}| \right\} \\ & - 8\beta\left(\alpha + \frac{\pi}{2}\right)|m_{ee}^{(2)}||m_{ee}^{(3)}|, \end{aligned} \quad (85)$$

or

$$\begin{pmatrix} \beta' & \alpha' \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} \approx K, \quad (86)$$

where

$$\begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha + \frac{\pi}{2} \end{pmatrix}, \quad (87)$$

$$\begin{aligned} a &= -(|m_{ee}^{(1)}| - |m_{ee}^{(2)}|)|m_{ee}^{(3)}| > 0, \\ b &= -|m_{ee}^{(2)}||m_{ee}^{(3)}| < 0, \\ c &= (|m_{ee}^{(1)}| + |m_{ee}^{(3)}|)|m_{ee}^{(2)}| > 0, \\ K &= \frac{1}{4} \left\{ |m_{ee}|^2 - |m_{ee}|_{\min}^2 \right\} > 0, \end{aligned} \quad (88)$$

and then $ac - b^2 > 0$ and $c - a > 0$. As in section 3.1, Eq.(86) can be rewritten as

$$\begin{pmatrix} \beta'' & \alpha'' \end{pmatrix} \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} \approx K, \quad (89)$$

where

$$\begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \beta \\ \alpha + \frac{\pi}{2} \end{pmatrix}, \quad (90)$$

and

$$\begin{aligned} \tan \theta &= \frac{-1}{2|m_{ee}^{(2)}||m_{ee}^{(3)}|} \left[|m_{ee}^{(1)}|(|m_{ee}^{(2)}| + |m_{ee}^{(3)}|) \right. \\ &\quad \left. - \sqrt{|m_{ee}^{(1)}|^2(|m_{ee}^{(3)}| + |m_{ee}^{(2)}|)^2 + 4(|m_{ee}^{(2)}||m_{ee}^{(3)}|)^2} \right] > 0, \\ \tan 2\theta &= \frac{2|m_{ee}^{(2)}||m_{ee}^{(3)}|}{|m_{ee}^{(1)}|(|m_{ee}^{(2)}| + |m_{ee}^{(3)}|)} > 0, \\ \lambda_{\pm} &= \frac{1}{2} \left[\left\{ -|m_{ee}^{(1)}||m_{ee}^{(3)}| + 2|m_{ee}^{(2)}||m_{ee}^{(3)}| + |m_{ee}^{(1)}||m_{ee}^{(2)}| \right\} \right. \\ &\quad \left. \pm \sqrt{|m_{ee}^{(1)}|^2(|m_{ee}^{(3)}| + |m_{ee}^{(2)}|)^2 + 4(|m_{ee}^{(2)}||m_{ee}^{(3)}|)^2} \right]. \end{aligned} \quad (91)$$

Because of $\tan \theta > 0$ and $\tan 2\theta > 0$, one has $0 < \theta < \pi/4$. In the region (C), $|m_{ee}^{(1)}| < |m_{ee}^{(2)}| - |m_{ee}^{(3)}|$, the isocontour around the point of the minimum, $|m_{ee}|_{\min} = |m_{ee}^{(2)}| - |m_{ee}^{(3)}| - |m_{ee}^{(1)}|$ becomes the ellipse represented by the following equation,

$$\frac{\beta'^2}{\left(\sqrt{\frac{K}{\lambda_-}}\right)^2} + \frac{\alpha'^2}{\left(\sqrt{\frac{K}{\lambda_+}}\right)^2} = 1, \quad (92)$$

where the fourth order of two variables β and $(\alpha + \pi/2)$ or higher orders have been neglected. This ellipse has a center at $(\beta, \alpha) = (0, -\pi/2)$. The direction of the β'' axis is produced by a counterclockwise rotation of the β axis by the angle $\theta(> 0)$.

Although the expressions of λ_{\pm} or $\tan \theta$ are very complicated, these expressions become simple when the value of the lightest neutrino mass m_1 ($|m_{ee}^{(1)}| = m_1|U_{e1}|^2$) satisfies,

$$|m_{ee}^{(3)}| \gg |m_{ee}^{(1)}|. \quad (93)$$

Owing to this inequality, we can neglect the terms of order $O(|m_{ee}^{(1)}|^2)$, then the angle θ and the ratio of the minor axis to the major axis $\sqrt{\lambda_-/\lambda_+}$ become

$$\begin{aligned} \theta &\approx \frac{\pi}{4} - \frac{(|m_{ee}^{(2)}| + |m_{ee}^{(3)}|)}{4|m_{ee}^{(2)}||m_{ee}^{(3)}|}|m_{ee}^{(1)}|, \\ \sqrt{\frac{\lambda_-}{\lambda_+}} &\approx \sqrt{\frac{(|m_{ee}^{(2)}| - |m_{ee}^{(3)}|)}{4|m_{ee}^{(2)}||m_{ee}^{(3)}|}|m_{ee}^{(1)}|}. \end{aligned} \quad (94)$$

In the region where the condition Eq.(93) is satisfied, the angle θ increases and the ratio $\sqrt{\lambda_-/\lambda_+}$ decreases when the value of the lightest neutrino mass m_1 is decreased. When the size of the major axis of the ellipse is $\pi/4$, the effective Majorana mass becomes

$$|m_{ee}| \approx |m_{ee}|_{\min} + \frac{\pi^2}{64}|m_{ee}^{(1)}|. \quad (95)$$

In the $m_1 \rightarrow 0$ limit, we have

$$\theta \rightarrow \frac{\pi}{4}, \quad \sqrt{\frac{\lambda_-}{\lambda_+}} \rightarrow 0, \quad (m_1 \rightarrow 0). \quad (96)$$

These behavior in the $m_1 \rightarrow 0$ limit resembles those of Eq.(46) in the case of the ellipse representing the isocontour around the point of the maximum $|m_{ee}|$.

3.3 Region except extremal values of $|m_{ee}|$.

In the preceding subsections, we have obtained the isocontour around the point of the maximum or minimum $|m_{ee}|$ in the $\beta\alpha$ -plane. What are the isocontours not around the point of the maximum or minimum $|m_{ee}|$? We shall obtain such isocontours in

this subsection. In the normal mass ordering case, the $|m_{ee}^{(1)}|$ changes from zero to the maximum of $|m_{ee}^{(i)}|$ ($i = 1, 2, 3$) when the value of the lightest neutrino mass m_1 is varied. If the values of m_1 (or $|m_{ee}^{(1)}| = m_1|U_{e1}|^2$) is in some regions, one can obtain the approximated equations analytically which represent the isocontours not around the point of $|m_{ee}|_{\max}$ or $|m_{ee}|_{\min}$. In the following, we give such three regions of $|m_{ee}^{(1)}|$ and derive the approximated equation representing the isocontour in each three cases.

3.3.1 The first case: $|m_{ee}^{(1)}| > |m_{ee}^{(2)}| \gg |m_{ee}^{(3)}|$.

The first case is that the value of the lightest neutrino mass m_1 ($= |m_{ee}^{(1)}|/|U_{e1}|^2$) is large enough to satisfy the following relation,

$$|m_{ee}^{(1)}| > |m_{ee}^{(2)}| \gg |m_{ee}^{(3)}|. \quad (97)$$

In order to satisfy this relation, m_1^2 should be much larger than Δm_\odot^2 , $m_1^2 \gg \Delta m_\odot^2$. In this first case, $|m_{ee}^{(3)}|$ is much smaller than $|m_{ee}^{(1)}|$ and $|m_{ee}^{(2)}|$. At first, let $|m_{ee}^{(3)}|$ equal zero and the Majorana phase α_0 satisfies, $|m_{ee}| = \left| |m_{ee}^{(1)}| + |m_{ee}^{(2)}|e^{2i\alpha_0} \right|$, or

$$\cos(2\alpha_0) = \frac{|m_{ee}|^2 - |m_{ee}^{(1)}|^2 - |m_{ee}^{(2)}|^2}{2|m_{ee}^{(1)}||m_{ee}^{(2)}|}. \quad (98)$$

The other phase β can not be determined at all. In practice the $|m_{ee}^{(3)}|$ is not zero, and we will obtain the isocontour of $|m_{ee}|$ by the method of perturbation with $|m_{ee}^{(3)}|$, or more formally with $|m_{ee}^{(3)}|/\sqrt{|m_{ee}^{(1)}||m_{ee}^{(2)}|}$. We rewrite

$$\begin{aligned} |m_{ee}| &= \left| |m_{ee}^{(1)}| + |m_{ee}^{(2)}|e^{2i\alpha} + |m_{ee}^{(3)}|e^{2i\beta} \right| \\ &\equiv \sqrt{|m_{ee}^{(1)}||m_{ee}^{(2)}|} |X_1 + X_2 e^{2i\alpha} + X_3 e^{2i\beta}|, \end{aligned} \quad (99)$$

or

$$\frac{|m_{ee}|^2}{|m_{ee}^{(1)}||m_{ee}^{(2)}|} = |X_1 + X_2 e^{2i\alpha} + X_3 e^{2i\beta}|^2, \quad (100)$$

where

$$X_1 \equiv \sqrt{\frac{|m_{ee}^{(1)}|}{|m_{ee}^{(2)}|}}, \quad X_2 \equiv \sqrt{\frac{|m_{ee}^{(2)}|}{|m_{ee}^{(1)}|}}, \quad X_3 \equiv \frac{|m_{ee}^{(3)}|}{\sqrt{|m_{ee}^{(1)}||m_{ee}^{(2)}|}}, \quad (101)$$

and $X_1 > X_2 \gg X_3$, $X_1 X_2 = 1$.

If $X_3 (\ll 1)$ in the right-handed side of Eq.(100) is neglected, the phase $\alpha = \alpha_0$ satisfies

$$\frac{|m_{ee}|^2}{|m_{ee}^{(1)}||m_{ee}^{(2)}|} = |X_1 + X_2 e^{2i\alpha_0}|^2, \quad (102)$$

or

$$\cos(2\alpha_0) = \frac{1}{2} \left[\frac{|m_{ee}|^2}{|m_{ee}^{(1)}||m_{ee}^{(2)}|} - X_1^2 - X_2^2 \right]. \quad (103)$$

Now, we seek the α by the perturbation with X_3 , and we expand α as

$$\alpha = \sum_{n=0}^{\infty} \alpha_n X_3^n = \alpha_0 + \alpha_1 X_3 + \alpha_2 X_3^2 + \cdots. \quad (104)$$

Then, it leads

$$\begin{aligned} & |X_1 + X_2 e^{2i\alpha} + X_3 e^{2i\beta}|^2 \\ &= |X_1 + X_2 e^{2i(\alpha_0 + \alpha_1 X_3 + \alpha_2 X_3^2 + \cdots)} + X_3 e^{2i\beta}|^2 \\ &= |a_0 + a_1 X_3 + a_2 X_3^2 + \cdots|^2, \end{aligned} \quad (105)$$

where

$$\begin{aligned} a_0 &\equiv X_1 + X_2 e^{2i\alpha_0}, \\ a_1 &\equiv 2i\alpha_1 X_2 e^{2i\alpha_0} + e^{2i\beta}, \\ a_2 &\equiv 2X_2 (i\alpha_2 - \alpha_1^2) e^{2i\alpha_0}, \end{aligned} \quad (106)$$

and so on. Owing to the definition of α_0 , the a_0 satisfies

$$\frac{|m_{ee}|^2}{|m_{ee}^{(1)}||m_{ee}^{(2)}|} = |a_0|^2. \quad (107)$$

When the order of X_3^2 is neglected in Eq(105), one has from Eq.(100),

$$\frac{|m_{ee}|^2}{|m_{ee}^{(1)}||m_{ee}^{(2)}|} = |a_0 + a_1 X_3 + O(X_3^2)|^2 = |a_0|^2 + 2\text{Re}(a_0^* a_1) X_3 + O(X_3^2). \quad (108)$$

The α_1 can be determined by

$$\text{Re}(a_0^* a_1) = 0, \quad (109)$$

and the result is

$$\begin{aligned} \alpha_1 &= \frac{1}{2 \sin(2\alpha_0)} \{(\text{Re } a_0) \cos(2\beta) + (\text{Im } a_0) \sin(2\beta)\} \\ &= \frac{1}{2 \sin(2\alpha_0)} \{|a_0| \cos(2\beta + \phi)\} \\ &= \frac{1}{2 \sin(2\alpha_0)} \sqrt{X_1^2 + X_2^2 + 2 \cos(2\alpha_0)} \cos(2\beta + \phi) \\ &= \frac{1}{2 \sin(2\alpha_0)} \frac{|m_{ee}|}{\sqrt{|m_{ee}^{(1)}||m_{ee}^{(2)}|}} \cos(2\beta + \phi), \end{aligned} \quad (110)$$

where

$$\tan \phi = -\frac{\text{Im } a_0}{\text{Re } a_0} = -\frac{X_2 \sin(2\alpha_0)}{X_1 + X_2 \cos(2\alpha_0)}. \quad (111)$$

In the same manner, the coefficient, α_2 of the second order of X_3^2 can be obtained,

$$\alpha_2 = \frac{1}{4 \sin(2\alpha_0)} \left\{ 1 - 4\alpha_1^2 \cos(2\alpha_0) + 4\alpha_1 X_2 \sin(2\beta - 2\alpha_0) \right\}. \quad (112)$$

In this expansion, it should be $\sin 2\alpha_0 \neq 0$. This shows that we can not use this expansion near $\alpha_0 = 0$ or $\alpha_0 = -\pi/2$. When $\alpha_0 \approx 0$, it becomes $\cos(2\alpha_0) \approx 1$, and $|m_{ee}| \approx |m_{ee}^{(1)}| + |m_{ee}^{(2)}| \approx |m_{ee}|_{\max}$. When $\alpha_0 \approx -\pi/2$, it becomes $\cos(2\alpha_0) \approx -1$, and $|m_{ee}| \approx \sqrt{|m_{ee}^{(1)}|^2 + |m_{ee}^{(2)}|^2} - 2|m_{ee}^{(1)}||m_{ee}^{(2)}| = |m_{ee}^{(1)}| - |m_{ee}^{(2)}| \approx |m_{ee}|_{\min}$. From these, we understand that the approximation using the expansion in X_3 does not work near the maximum value $|m_{ee}|_{\max}$ or the minimum value $|m_{ee}|_{\min}$.

Since X_3 is very small in this case, we only take the first order $O(X_3)$, and the relation between α and β becomes

$$\begin{aligned} \alpha &= \alpha_0 + X_3 \frac{1}{2 \sin(2\alpha_0)} \sqrt{X_1^2 + X_2^2 + 2 \cos(2\alpha_0)} \cos(2\beta + \phi) + O(X_3^2) \\ &= \alpha_0 + X_3 \frac{1}{2 \sin(2\alpha_0)} \frac{|m_{ee}|}{\sqrt{|m_{ee}^{(1)}||m_{ee}^{(2)}|}} \cos(2\beta + \phi) + O(X_3^2) \\ &= \alpha_0 + \left(\frac{m_3}{m_1 m_2} \right) \frac{|m_{ee}|}{2 \sin(2\alpha_0)} \frac{|U_{e3}|^2}{|U_{e1}|^2 |U_{e2}|^2} \cos(2\beta + \phi) + O(X_3^2). \end{aligned} \quad (113)$$

The α is the sum of the constant α_0 and the cosine function of β with the period π which function is multiplied by the small quantity.

For instance, consider the case that $\alpha_0 = -\pi/4$. From Eq.(98), the effective Majorana mass takes the value $|m_{ee}| = \sqrt{|m_{ee}^{(1)}|^2 + |m_{ee}^{(2)}|^2}$ and the equation of the isocontour of this $|m_{ee}|$ becomes

$$\alpha = -\frac{\pi}{4} - \frac{\sqrt{|m_{ee}^{(1)}|^2 + |m_{ee}^{(2)}|^2}}{2|m_{ee}^{(1)}||m_{ee}^{(2)}|} |m_{ee}^{(3)}| \cos(2\beta + \phi) + O(X_3^2), \quad (114)$$

where $\tan \phi = |m_{ee}^{(2)}|/|m_{ee}^{(1)}| < 1$. When the mass m_1 decreases, the factor $m_3/(m_1 m_2)$ in the coefficient of $\cos(2\beta + \phi)$ increases. Thereby, the contribution of the second term in the right-handed side of Eq.(114) becomes larger when the value of the lightest neutrino mass m_1 decreases. While the constraint Eq.(113) can not determine the value of β , it restricts the value of α within a rather narrow range. To see it quantitatively, the second term in the right-handed side of Eq.(113) is estimated as

$$\left| \frac{1}{2 \sin(2\alpha_0)} \frac{|m_{ee}^{(3)}|}{|m_{ee}^{(1)}||m_{ee}^{(2)}|} |m_{ee}| \cos(2\beta + \phi) \right| \leq \frac{1}{2 |\sin(2\alpha_0)|} \frac{|m_{ee}^{(3)}|}{|m_{ee}^{(1)}||m_{ee}^{(2)}|} |m_{ee}|. \quad (115)$$

For instance, again consider the case of $\alpha_0 = -\pi/4$ and we have

$$\left| \frac{1}{2 \sin(2\alpha_0)} \frac{|m_{ee}^{(3)}|}{|m_{ee}^{(1)}||m_{ee}^{(2)}|} |m_{ee}| \cos(2\beta + \phi) \right| \leq \frac{1}{2} \frac{\sqrt{|m_{ee}^{(1)}|^2 + |m_{ee}^{(2)}|^2}}{|m_{ee}^{(1)}||m_{ee}^{(2)}|} |m_{ee}^{(3)}|, \quad (116)$$

which is small because of $|m_{ee}^{(1)}| > |m_{ee}^{(2)}| \gg |m_{ee}^{(3)}|$. When the value of the lightest neutrino mass m_1 decreases, the range restricting the value of α enlarges.

3.3.2 The second case; $|m_{ee}^{(1)}| > |m_{ee}^{(2)}| > |m_{ee}^{(3)}|$.

The second case is that the lightest neutrino mass m_1 has the value satisfying

$$|m_{ee}^{(1)}| > |m_{ee}^{(2)}| > |m_{ee}^{(3)}|. \quad (117)$$

This value of m_1 is not so large as that in the first case (the proceeding sub-subsection). In order to satisfy the inequality, $|m_{ee}^{(1)}| > |m_{ee}^{(2)}|$, the m_1 should be

$$m_1^2 > \frac{\Delta m_\odot^2}{\frac{|U_{e1}|^4}{|U_{e2}|^4} - 1}. \quad (118)$$

If one puts $\sin^2 \theta_{12} = 0.30$ for the moment, it becomes

$$\frac{1}{\frac{|U_{e1}|^4}{|U_{e2}|^4} - 1} \approx 0.23. \quad (119)$$

Although $|m_{ee}^{(3)}|$ is smaller than $|m_{ee}^{(1)}|$ and $|m_{ee}^{(2)}|$, the X_3 in this second case is not so small as that in the first case. When we take into account the second order of X_3 in Eq.(113), the α is represented as

$$\alpha = \alpha_0 + X_3 \frac{1}{2 \sin(2\alpha_0)} \sqrt{X_1^2 + X_2^2 + 2 \cos(2\alpha_0)} \cos(2\beta + \phi) + \alpha_2 X_3^2 + O(X_3^3), \quad (120)$$

where the coefficient, α_2 , is from Eq.(112),

$$\begin{aligned} \alpha_2 &= \frac{1}{4 \sin(2\alpha_0)} \left\{ 1 - 4\alpha_1^2 \cos(2\alpha_0) + 4\alpha_1 X_2 \sin(2\beta - 2\alpha_0) \right\} \\ &= \frac{1}{4 \sin(2\alpha_0)} \left[1 - \frac{\cos(2\alpha_0) \{X_1^2 + X_2^2 + 2 \cos(2\alpha_0)\}}{2 \sin^2(2\alpha_0)} \{ \cos(4\beta + 2\phi) + 1 \} \right. \\ &\quad \left. + \frac{X_2 \sqrt{X_1^2 + X_2^2 + 2 \cos(2\alpha_0)}}{\sin(2\alpha_0)} \{ \sin(4\beta + \phi - 2\alpha_0) - \sin(\phi + 2\alpha_0) \} \right] \\ &= \frac{1}{4 \sin(2\alpha_0)} \left[1 - \frac{\cos(2\alpha_0)}{2 \sin^2(2\alpha_0)} \frac{|m_{ee}|^2}{|m_{ee}^{(1)}| |m_{ee}^{(2)}|} \{ \cos(4\beta + 2\phi) + 1 \} \right. \\ &\quad \left. + \frac{1}{\sin(2\alpha_0)} \frac{|m_{ee}|}{|m_{ee}^{(1)}|} \{ \sin(4\beta + \phi - 2\alpha_0) - \sin(\phi + 2\alpha_0) \} \right]. \quad (121) \end{aligned}$$

The coefficient α_1 , Eq.(110), involves the cosine function of β with the period π , and the coefficient α_2 involves the cosine (sine) function of β with the period $\pi/2$. The α is the sum of the constant α_0 , the term involving the cosine function of β with the period π , and the other term involving the cosine (sine) function of β with the period $\pi/2$. If needed, the higher orders of $O(X_3^n)$ can be calculated in the same manner.

3.3.3 The third case; $|m_{ee}^{(2)}| > |m_{ee}^{(3)}| \gg |m_{ee}^{(1)}|$.

The third case is that the value of the lightest neutrino mass m_1 ($= |m_{ee}^{(1)}|/|U_{e1}|^2$) is small enough to satisfy the following relation,

$$|m_{ee}^{(2)}| > |m_{ee}^{(3)}| \gg |m_{ee}^{(1)}|. \quad (122)$$

In this third case, $|m_{ee}^{(1)}|$ is much smaller than $|m_{ee}^{(2)}|$ and $|m_{ee}^{(3)}|$. At first, let $|m_{ee}^{(1)}|$ equal zero, and the difference of two Majorana phases $\alpha - \beta$ satisfies,

$$|m_{ee}| = |m_{ee}^{(2)}|e^{2i\alpha} + |m_{ee}^{(3)}|e^{2i\beta} = |m_{ee}^{(2)}|e^{2i(\alpha-\beta)} + |m_{ee}^{(3)}|, \quad (123)$$

or [18]

$$\cos(2(\alpha - \beta)) = \frac{|m_{ee}|^2 - |m_{ee}^{(2)}|^2 - |m_{ee}^{(3)}|^2}{2|m_{ee}^{(2)}||m_{ee}^{(3)}|}. \quad (124)$$

In Ref.[18], the dependence of the effective Majorana mass $|m_{ee}|$ on $\alpha - \beta$, Eq.(124), was studied in detail. Defining this difference $\alpha - \beta$ as γ_0 , we have

$$\alpha = \gamma_0 + \beta, \quad (125)$$

and

$$\cos(2\gamma_0) = \frac{|m_{ee}|^2 - |m_{ee}^{(2)}|^2 - |m_{ee}^{(3)}|^2}{2|m_{ee}^{(2)}||m_{ee}^{(3)}|}. \quad (126)$$

In practice, the $|m_{ee}^{(1)}|$ is not zero, and we will obtain the isocontour of $|m_{ee}|$ by the method of perturbation with $|m_{ee}^{(1)}|$, or more formally with $|m_{ee}^{(1)}|/\sqrt{|m_{ee}^{(3)}||m_{ee}^{(2)}|} \equiv Y_1$, which is discussed in Appendix. When one takes up to the first order of Y_1 , the relation between α and β becomes (see Eq.(A.16) in Appendix),

$$\begin{aligned} \alpha &= \gamma_0 + \beta + Y_1 \frac{1}{2 \sin(2\gamma_0)} \sqrt{Y_3^2 + Y_2^2 + 2 \cos(2\gamma_0)} \cos(2\beta - \varphi) + O(Y_1^2) \\ &= \gamma_0 + \beta + \left(\frac{m_1}{m_3 m_2} \right) \frac{|m_{ee}|}{2 \sin(2\gamma_0)} \frac{|U_{e1}|^2}{|U_{e3}|^2 |U_{e2}|^2} \cos(2\beta - \varphi) + O(Y_1^2), \end{aligned} \quad (127)$$

where

$$Y_2 \equiv \sqrt{\frac{|m_{ee}^{(2)}|}{|m_{ee}^{(3)}|}}, \quad Y_3 \equiv \sqrt{\frac{|m_{ee}^{(3)}|}{|m_{ee}^{(2)}|}}, \quad \tan \varphi = -\frac{Y_2 \sin(2\gamma_0)}{Y_3 + Y_2 \cos(2\gamma_0)}. \quad (128)$$

The α is the sum of the constant γ_0 , the β , and the cosine function of β with the period π which function is multiplied by the small quantity. As discussed in Appendix, the approximation using the expansion in Y_1 does not work near the maximum value $|m_{ee}|_{\max}$ or the minimum value $|m_{ee}|_{\min}$.

In subsection 3.3, we have considered the three cases and developed the appropriate method of perturbation in each case. However, the case, $|m_{ee}^{(2)}| > |m_{ee}^{(1)}| \gtrsim |m_{ee}^{(3)}|$, has not been discussed. Unfortunately, we are not able to find a useful method of approximation in this case.

4 Majorana phases for the inverted mass ordering

In this chapter, we derive the equation representing the isocontour of the effective Majorana mass $|m_{ee}|$ in the $\beta\alpha$ -plane in the inverted mass ordering case ($m_2 > m_1 > m_3$). As discussed in section 2, the lightest neutrino mass m_3 is treated as a free parameter, and when we set the value of m_3 , the effective Majorana mass $|m_{ee}|$ is considered as a function of β and α . If the value of $|m_{ee}|$ is determined, the constraint on β and α is obtained, yet we can not obtain both values of β and α simultaneously as in the case of the normal mass ordering. In the inverted mass ordering case, the order of three $|m_{ee}^{(i)}|$ is distinctly known for an arbitrary value of m_3 ,

$$|m_{ee}^{(1)}| > |m_{ee}^{(2)}| \gg |m_{ee}^{(3)}|, \quad m_1 \approx m_2. \quad (129)$$

Hence, we can discuss about the isocontour of $|m_{ee}|$ in short compared to the normal mass ordering case. In the same manner as the normal mass ordering case, we first obtain the isocontour of $|m_{ee}|$ around the point of maximum $|m_{ee}|$ in the $\beta\alpha$ -plane, and next those around the point of minimum $|m_{ee}|$, restricting the region of β and α as Eq.(21). In the final subsection, we obtain the isocontour except around the point of maximum or minimum $|m_{ee}|$.

4.1 Around maximum value of $|m_{ee}|$.

For an arbitrary value of the lightest neutrino mass m_3 , the effective Majorana mass $|m_{ee}|$ takes a maximum value,

$$|m_{ee}|_{\max} = |m_{ee}^{(1)}| + |m_{ee}^{(2)}| + |m_{ee}^{(3)}|, \quad (130)$$

at the point $(\beta, \alpha) = (0, 0)$ in the $\beta\alpha$ -plane. Let us expand $|m_{ee}|^2$ around this point. Including the terms up to second order on the two variables β and α in the expansion, we have the same expressions as those from Eq.(24) to Eq.(32) in section 3. If one neglects the fourth order of two variables β and α or higher orders, the isocontour around the point of maximum $|m_{ee}|_{\max}$ becomes the ellipse represented by

$$\frac{\beta''^2}{\left(\sqrt{\frac{J}{\lambda_-}}\right)^2} + \frac{\alpha''^2}{\left(\sqrt{\frac{J}{\lambda_+}}\right)^2} = 1, \quad (131)$$

where

$$\begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix}, \quad (132)$$

and

$$J = \frac{1}{4} \left\{ |m_{ee}|_{\max}^2 - |m_{ee}|^2 \right\} > 0. \quad (133)$$

The direction of the β'' axis is produced by a counterclockwise rotation of the β axis by the angle $\theta(> 0)$.

Since $|m_{ee}^{(3)}|$ is much smaller than $|m_{ee}^{(1)}|$ and $|m_{ee}^{(2)}|$, one can neglect the terms of order $O(|m_{ee}^{(3)}|^2)$ in the expressions of Eq.(28) in the inverted mass ordering case. The $\tan \theta$ and the ratio of the minor axis to the major axis, $\sqrt{\lambda_-/\lambda_+}$, are approximated, respectively,

$$\tan \theta \approx \frac{|m_{ee}^{(3)}|}{|m_{ee}^{(1)}|} = \frac{m_3}{m_1} \frac{|U_{e3}|^2}{|U_{e1}|^2} \ll 1,$$

$$\sqrt{\frac{\lambda_-}{\lambda_+}} \approx \sqrt{\frac{(|m_{ee}^{(1)}| + |m_{ee}^{(2)}|)|m_{ee}^{(3)}|}{|m_{ee}^{(1)}||m_{ee}^{(2)}|}} \approx \sqrt{\frac{m_3}{m_1}} \sqrt{\frac{|U_{e3}|^2(1 - |U_{e3}|^2)}{|U_{e1}|^2|U_{e2}|^2}}, \quad (134)$$

where we have used $m_1 \approx m_2$. We should notice here that, in the inverted (normal) mass ordering case, the lightest neutrino mass m_3 (m_1) is regarded as a free parameter. When the value of the lightest neutrino mass m_3 is decreased, both the angle θ and the ratio $\sqrt{\lambda_-/\lambda_+}$ decrease monotonously. In the $m_3 \rightarrow 0$ limit, we have

$$\theta \rightarrow 0, \quad \sqrt{\frac{\lambda_-}{\lambda_+}} \rightarrow 0, \quad (m_3 \rightarrow 0). \quad (135)$$

The behavior of the angle θ and the ratio $\sqrt{\lambda_-/\lambda_+}$ by decreasing the value of the lightest neutrino mass in this subsection of the inverted mass ordering case is strikingly different from that in the subsection 3.1 of the normal mass ordering case. When the size of the major axis of the ellipse is a quarter of π , $2\sqrt{J/\lambda_-} = \pi/4$, the effective Majorana mass becomes

$$|m_{ee}| \approx |m_{ee}|_{\max} - \frac{\pi^2}{32} |m_{ee}^{(3)}|. \quad (136)$$

4.2 Around minimum value of $|m_{ee}|$.

Owing to the relation Eq.(129) in the inverted mass ordering case, the effective Majorana mass takes a minimum value,

$$|m_{ee}|_{\min} = |m_{ee}^{(1)}| - |m_{ee}^{(2)}| - |m_{ee}^{(3)}|, \quad (137)$$

at the point $(\beta, \alpha) = (\pi/2, -\pi/2)$. Let us expand $|m_{ee}|^2$ around this point. Including the terms up to second order in the expansion, we have the same expressions as those from Eq.(49) to Eq.(56) in section 3. If one disregards the fourth order or higher orders, the isocontour around the point of minimum $|m_{ee}|_{\min}$ becomes the ellipse represented by

$$\frac{\beta'^2}{\left(\sqrt{\frac{K}{\lambda_-}}\right)^2} + \frac{\alpha'^2}{\left(\sqrt{\frac{K}{\lambda_+}}\right)^2} = 1, \quad (138)$$

where

$$\begin{pmatrix} \beta'' \\ \alpha'' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \beta' \\ \alpha' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} (\beta - \frac{\pi}{2}) \\ (\alpha + \frac{\pi}{2}) \end{pmatrix}, \quad (139)$$

and

$$K = \frac{1}{4} \left\{ |m_{ee}|^2 - |m_{ee}|_{\min}^2 \right\} > 0. \quad (140)$$

This ellipse has the center at $(\beta, \alpha) = (\pi/2, -\pi/2)$, and the direction of the β'' axis is produced by a clockwise rotation of the β axis by the angle $|\theta| (> 0)$.

Because one can neglect the terms of order $O(|m_{ee}^{(3)}|^2)$ in the expressions of Eq.(55) in the inverted mass ordering case, the angle θ and the ratio of the minor axis to the major axis, $\sqrt{\lambda_-/\lambda_+}$, are approximated, respectively,

$$\begin{aligned} \theta &\approx -\frac{|m_{ee}^{(3)}|}{|m_{ee}^{(1)}|} = -\frac{m_3}{m_1} \frac{|U_{e3}|^2}{|U_{e1}|^2} < 0, \quad |\theta| \ll 1, \\ \sqrt{\frac{\lambda_-}{\lambda_+}} &\approx \sqrt{\frac{(|m_{ee}^{(1)}| - |m_{ee}^{(2)}|)|m_{ee}^{(3)}|}{|m_{ee}^{(1)}||m_{ee}^{(2)}|}} \approx \sqrt{\frac{m_3}{m_1}} \sqrt{\frac{|U_{e3}|^2(|U_{e1}|^2 - |U_{e2}|^2)}{|U_{e1}|^2|U_{e2}|^2}}, \end{aligned} \quad (141)$$

where we have used $m_1 \approx m_2$. When the value of the lightest neutrino mass m_3 is decreased, both the absolute value of the angle $|\theta|$ and the ratio $\sqrt{\lambda_-/\lambda_+}$ decrease monotonously. In the $m_3 \rightarrow 0$ limit, one has

$$\theta \rightarrow 0, \quad \sqrt{\frac{\lambda_-}{\lambda_+}} \rightarrow 0, \quad (m_3 \rightarrow 0). \quad (142)$$

The behavior of the angle θ and the ration $\sqrt{\lambda_-/\lambda_+}$ by decreasing the value of the lightest neutrino mass in this subsection of the inverted mass ordering case is strikingly different from that in the sub-subsection 3.2.1 of the normal mass ordering case. When the size of the major axis of the ellipse is $\pi/4$, the effective Majorana mass becomes

$$|m_{ee}| \approx |m_{ee}|_{\min} + \frac{\pi^2}{32} |m_{ee}^{(3)}| = |m_{ee}^{(1)}| - |m_{ee}^{(2)}| - \left(1 - \frac{\pi^2}{32}\right) |m_{ee}^{(3)}|. \quad (143)$$

4.3 Region except extremal values of $|m_{ee}|$.

In this subsection, we will find the isocontour not around the point of the maximum or minimum $|m_{ee}|$. Since the relation $|m_{ee}^{(1)}| > |m_{ee}^{(2)}| \gg |m_{ee}^{(3)}|$ holds for an arbitrary value of the lightest neutrino mass m_3 in the inverted mass ordering case, we can use the method of perturbation with $|m_{ee}^{(3)}|/\sqrt{|m_{ee}^{(1)}||m_{ee}^{(2)}|} = X_3 \ll 1$, which is used in section 3.3.1. The relation between α and β is, from Eq.(113),

$$\begin{aligned} \alpha &= \alpha_0 + X_3 \frac{1}{2 \sin(2\alpha_0)} \frac{|m_{ee}|}{\sqrt{|m_{ee}^{(1)}||m_{ee}^{(2)}|}} \cos(2\beta + \phi) + O(X_3^2) \\ &= \alpha_0 + \left(\frac{m_3}{m_1 m_2}\right) \frac{|m_{ee}|}{2 \sin(2\alpha_0)} \frac{|U_{e3}|^2}{|U_{e1}|^2 |U_{e2}|^2} \cos(2\beta + \phi) + O(X_3^2), \end{aligned} \quad (144)$$

where

$$\tan \phi = -\frac{X_2 \sin(2\alpha_0)}{X_1 + X_2 \cos(2\alpha_0)}, \quad X_1 = \sqrt{\frac{|m_{ee}^{(1)}|}{|m_{ee}^{(2)}|}}, \quad X_2 = \sqrt{\frac{|m_{ee}^{(2)}|}{|m_{ee}^{(1)}|}}. \quad (145)$$

The α_0 is given by [18]

$$\cos(2\alpha_0) = \frac{|m_{ee}|^2 - |m_{ee}^{(1)}|^2 - |m_{ee}^{(2)}|^2}{2|m_{ee}^{(1)}||m_{ee}^{(2)}|}. \quad (146)$$

Note that this perturbative calculation can not be used near the point of the maximum $|m_{ee}|_{\max}$ or the minimum $|m_{ee}|_{\min}$ as discussed in section 3.3.1.

When the mass m_3 decreases, the factor $m_3/(m_1 m_2)$ in the coefficient of $\cos(2\beta + \phi)$ in Eq.(144) decreases. Hence, the contribution of the second term in the right-handed side of Eq.(144) becomes small when the value of the lightest neutrino mass m_3 decreases. This behavior of the second term in the right-handed side of Eq.(144) by decreasing the value of the lightest neutrino mass in this subsection of the inverted mass ordering case is strikingly different from that in sub-subsection 3.3.1 of the normal mass ordering case. If the $m_3 \rightarrow 0$ limit is taken, one has

$$\alpha \rightarrow \alpha_0, \quad (m_3 \rightarrow 0), \quad (147)$$

and α approaches the constant not depending on β [18].

5 Conclusion.

In the $\beta\alpha$ -plane where β and α are the Majorana CP-violating phases, we obtained analytically the equation representing the isocontour of the effective Majorana mass $|m_{ee}|$ by the method of perturbation. The effective Majorana mass $|m_{ee}|$ is written as

$$|m_{ee}| = \left| |m_{ee}^{(1)}| + |m_{ee}^{(2)}|e^{2i\alpha} + |m_{ee}^{(3)}|e^{2i\beta} \right|, \quad (148)$$

where $|m_{ee}^{(1)}| = m_1|U_{e1}|^2$, $|m_{ee}^{(2)}| = m_2|U_{e2}|^2$, and $|m_{ee}^{(3)}| = m_3|U_{e3}|^2$. The equation representing the isocontour of $|m_{ee}|$ is expressed by the following six quantities; $|m_{ee}|$, two lepton mixing angles $(\theta_{12}, \theta_{13})$, two neutrino mass squared differences Δm_{\odot}^2 (responsible for the solar neutrino oscillation) and Δm_A^2 (responsible for the atmospheric neutrino oscillation), and the lightest neutrino mass which is regarded as a free parameter. We studied how the isocontour of $|m_{ee}|$ changes in the $\beta\alpha$ -plane when the value of the lightest neutrino mass is varied in the case of the normal mass ordering and the inverted mass ordering, respectively.

In the normal mass ordering case ($m_3 > m_2 > m_1$), we assumed the relation $|m_{ee}^{(2)}| > |m_{ee}^{(3)}|$. When the value of the lightest neutrino mass m_1 is varied, the $|m_{ee}^{(1)}|$ changes from zero to the maximum of $|m_{ee}^{(i)}| (i = 1, 2, 3)$. The effective Majorana mass $|m_{ee}|$ has the maximum value at the point $(\beta, \alpha) = (0, 0)$, and the isocontour of $|m_{ee}|$ around this

point is the ellipse whose major axis is tilted in the counterclockwise direction of the β axis on the $\beta\alpha$ -plane. When the value of the lightest neutrino mass m_1 is varied, we examined how the form of the ellipse or the direction of the major axis changes in the $\beta\alpha$ -plane as discussed in section 3.1. In finding the minimum value of $|m_{ee}|$, it is convenient to divide into the following three regions according to the size of $|m_{ee}^{(1)}|$,

- region (A); $|m_{ee}^{(2)}| + |m_{ee}^{(3)}| < |m_{ee}^{(1)}|$,
- region (B); $|m_{ee}^{(2)}| - |m_{ee}^{(3)}| < |m_{ee}^{(1)}| < |m_{ee}^{(2)}| + |m_{ee}^{(3)}|$,
- region (C); $|m_{ee}^{(1)}| < |m_{ee}^{(2)}| - |m_{ee}^{(3)}|$.

In the region (A), the $|m_{ee}|$ has the minimum value at the point $(\beta, \alpha) = (\pi/2, -\pi/2)$. The isocontour of $|m_{ee}|$ around this point is the ellipse whose major axis is in the clockwise direction of the β axis. On the boundary between the region (A) and region (B), $|m_{ee}^{(1)}| = |m_{ee}^{(2)}| + |m_{ee}^{(3)}|$, the effective Majorana mass $|m_{ee}|$ takes a minimum value at the point $(\beta, \alpha) = (\pi/2, -\pi/2)$. Around this point, the isocontour of $|m_{ee}|$ becomes the straight line within the approximation up to the second order. If one takes into account the higher order terms such as $O(\beta - \pi/2)^4$, etc., the isocontour would become a closed line. In the region (B), the $|m_{ee}|$ has the minimum value zero at the point (β_c, α_c) where $0 < \beta_c < \pi/2$ and $-\pi/2 < \alpha_c < -\pi/4$. The isocontour around this point is the ellipse. On the boundary between the region (B) and region (C), $|m_{ee}^{(1)}| = |m_{ee}^{(2)}| - |m_{ee}^{(3)}|$, the $|m_{ee}|$ has the minimum value at the point $(\beta, \alpha) = (0, -\pi/2)$. Around this point, the isocontour of $|m_{ee}|$ becomes the straight line within the approximation up to the second order. If we take into account the higher order terms such as $O(\beta^4)$, etc., the isocontour would become a closed line. In the region (C), the $|m_{ee}|$ takes the minimum value at the point $(\beta, \alpha) = (0, -\pi/2)$. The isocontour of $|m_{ee}|$ around this point is the ellipse whose major axis is in the counterclockwise direction of the β axis. We also found the isocontour not around the point of the maximum or minimum $|m_{ee}|$ in the $\beta\alpha$ -plane. The equation representing such isocontour is obtained analytically using the approximation method in the following three cases, (a)~(c).

- (a) $|m_{ee}^{(1)}| > |m_{ee}^{(2)}| \gg |m_{ee}^{(3)}|$.

In this case, the α is the sum of the constant term not depending on β , and the term of the cosine function of β with the period π which function is multiplied by the small quantity. When the value of the lightest neutrino mass m_1 is decreased, the contribution of the term involving the cosine function of β becomes larger. While the value of β can not be determined at all, the value of α can be restricted within a rather narrow range.

- (b) $|m_{ee}^{(1)}| > |m_{ee}^{(2)}| > |m_{ee}^{(3)}|$.

The α is the sum of the constant term not depending on β , the term proportional to the cosine function of β with the period π , and the term proportional to the cosine (sine) function of β with the period $\pi/2$. While the value of β can not be determined at all, the value of α can be restricted within a certain range. When the value of the lightest neutrino mass m_1 is decreased, the range restricting the value of α widens.

- (c) $|m_{ee}^{(2)}| > |m_{ee}^{(3)}| \gg |m_{ee}^{(1)}|$.

The α is the sum of the constant term not depending on β , the β , and the term proportional to the cosine function of β with the period π . When the value of the lightest neutrino mass m_1 is decreased, the term involving the cosine function of β approaches zero. Both the value of β and that of α can be restricted within certain ranges, respectively.

A thing which has not been discussed is the case, $|m_{ee}^{(2)}| > |m_{ee}^{(1)}| \gtrsim |m_{ee}^{(3)}|$, where the orders of $|m_{ee}^{(1)}|$, $|m_{ee}^{(2)}|$, and $|m_{ee}^{(3)}|$ are equal in magnitude. Unfortunately, we can not find a useful method of approximation in this case.

In the inverted mass ordering case ($m_2 > m_1 > m_3$), the following relation holds for an arbitrary value of the lightest neutrino mass m_3 ,

$$|m_{ee}^{(1)}| > |m_{ee}^{(2)}| >> |m_{ee}^{(3)}|, \quad m_1 \approx m_2. \quad (149)$$

The effective Majorana mass $|m_{ee}|$ takes the maximum value at the point $(\beta, \alpha) = (0, 0)$, and the isocontour of $|m_{ee}|$ around this point is the ellipse whose major axis is slightly tilted in the counterclockwise direction of the β axis on the $\beta\alpha$ -plane. When the value of the lightest neutrino mass m_3 is decreased, the major axis of the ellipse approaches the β axis, and the ratio of the minor axis of the ellipse to the major axis approaches zero. The $|m_{ee}|$ has the minimum value at the point $(\beta, \alpha) = (\pi/2, -\pi/2)$, and the isocontour of $|m_{ee}|$ around this point is the ellipse whose major axis is slightly tilted in the clockwise direction of the β axis on the $\beta\alpha$ -plane. When the value of the lightest neutrino mass m_3 is decreased, the major axis of the ellipse approaches the β axis, and the ratio of the minor axis of the ellipse to the major axis approaches zero. We also found the isocontour not around the point of the maximum or minimum $|m_{ee}|$ in the $\beta\alpha$ -plane. The equation representing that isocontour is such that the α is the sum of the constant term not depending on β , and the term of the cosine function of β with the period π which function is multiplied by the small quantity. When the value of the lightest neutrino mass m_3 is decreased, the contribution of the term involving the cosine function of β becomes small. While the value of β can not be determined at all, the value of α can be restricted within a very narrow range.

So far the constraints between the Majorana CP-violating phases α and β have been investigated mainly by the numerical calculations. At present, we can not decide whether the mass of the neutrino obeys the normal mass ordering or the inverted mass ordering, and the absolute neutrino mass scale is unknown. Under such circumstances remaining numerous possibilities, we would like to see prospectively the constraints between α and β (the isocontour of $|m_{ee}|$ in the $\beta\alpha$ -plane). In this paper, we obtained the equation of the constraint between α and β analytically by the method of perturbation, and clarified how the constraint changes in the $\beta\alpha$ -plane when the value of the lightest neutrino mass is varied. Since the perturbative calculations are possible up to any order, one can analytically calculate according to the required accuracy in principle.

A positive signal of the neutrinoless double beta decay has not been detected in the experiments, and the upper limit of the effective Majorana mass $|m_{ee}|$ has been

reported. The KamLAND-Zen experiment provides the upper limit [9],

$$|m_{ee}| < (120 - 250) \text{ meV} \quad \text{at } 90\% \text{ C.L.} \quad (150)$$

We expect that the upper limit of $|m_{ee}|$ will be lowered by the future neutrinoless double beta decay experiments. When the upper limit of $|m_{ee}|$ is put at a certain value, one can exclude some parameter regions in the $\beta\alpha$ -plane for a given value of the lightest neutrino mass. As discussed in section 2, we regard the other four parameters, $\theta_{12}, \theta_{13}, \Delta m_{\odot}^2$, and Δm_{A}^2 as given quantities in this paper. Let us assume that the value of the upper limit of $|m_{ee}|$ becomes so small that the inverted mass ordering is denied [26]. In the normal mass ordering, we consider here the case that the lightest neutrino mass m_1 satisfies the relation, $|m_{ee}^{(1)}| > |m_{ee}^{(2)}| >> |m_{ee}^{(3)}|$. If the upper limit of $|m_{ee}|$ becomes

$$|m_{ee}| \leq |m_{ee}|_{\text{max}} - \frac{\pi^2}{32} |m_{ee}^{(3)}| = |m_{ee}^{(1)}| + |m_{ee}^{(2)}| + \left(1 - \frac{\pi^2}{32}\right) |m_{ee}^{(3)}|, \quad (151)$$

then the parameter region inside the ellipse, Eq.(38),

$$\frac{\beta'^2}{\left(\frac{\pi}{8}\right)^2} + \frac{\alpha'^2}{\left(\frac{\pi}{8} \sqrt{\frac{(|m_{ee}^{(1)}| + |m_{ee}^{(2)}|)}{|m_{ee}^{(1)}| |m_{ee}^{(2)}|}} |m_{ee}^{(3)}|\right)^2} = 1, \quad (152)$$

is excluded in the $\beta\alpha$ -plane. Furthermore, if the upper limit of $|m_{ee}|$ is lowered as

$$|m_{ee}| \leq \sqrt{|m_{ee}^{(1)}|^2 + |m_{ee}^{(2)}|^2}, \quad (153)$$

then, from Eq.(98) and Eq.(114), the following parameter region,

$$\begin{aligned} & -\frac{\pi}{4} - \frac{\sqrt{|m_{ee}^{(1)}|^2 + |m_{ee}^{(2)}|^2}}{2|m_{ee}^{(1)}| |m_{ee}^{(2)}|} |m_{ee}^{(3)}| \cos(2\beta + \phi) \leq \alpha \\ & \leq \frac{\pi}{4} + \frac{\sqrt{|m_{ee}^{(1)}|^2 + |m_{ee}^{(2)}|^2}}{2|m_{ee}^{(1)}| |m_{ee}^{(2)}|} |m_{ee}^{(3)}| \cos(-2\beta + \phi), \end{aligned} \quad (154)$$

is excluded. In the future, if the neutrinoless double beta decay is detected by experiments and we can confirm the neutrino is Majorana fermion, a next challenge is to restrict the values of the Majorana phases α and β .

Appendix

Expansion in Y_1

The effective Majorana mass $|m_{ee}|$ is written as

$$|m_{ee}| = ||m_{ee}^{(1)}| + |m_{ee}^{(2)}|e^{2i\alpha} + |m_{ee}^{(3)}|e^{2i\beta}|. \quad (\text{A.1})$$

When the value of $|m_{ee}|$ is given, we will obtain the constraint equation between α and β under the following condition,

$$|m_{ee}^{(2)}| > |m_{ee}^{(3)}| \gg |m_{ee}^{(1)}|. \quad (\text{A.2})$$

The effective mass $|m_{ee}|$ can be rewritten as

$$\begin{aligned} |m_{ee}| &= ||m_{ee}^{(3)}| + |m_{ee}^{(2)}|e^{2i\gamma} + |m_{ee}^{(1)}|e^{-2i\beta}| \\ &\equiv \sqrt{|m_{ee}^{(3)}||m_{ee}^{(2)}|} |Y_3 + Y_2 e^{2i\gamma} + Y_1 e^{-2i\beta}|, \end{aligned} \quad (\text{A.3})$$

where $\gamma \equiv \alpha - \beta$ and

$$Y_3 \equiv \sqrt{\frac{|m_{ee}^{(3)}|}{|m_{ee}^{(2)}|}}, \quad Y_2 \equiv \sqrt{\frac{|m_{ee}^{(2)}|}{|m_{ee}^{(3)}|}}, \quad Y_1 \equiv \frac{|m_{ee}^{(1)}|}{\sqrt{|m_{ee}^{(3)}||m_{ee}^{(2)}|}}. \quad (\text{A.4})$$

They satisfy, $1 \gg Y_1$, $Y_2 > Y_3 \gg Y_1$, $Y_3 Y_2 = 1$, and

$$\frac{|m_{ee}|^2}{|m_{ee}^{(3)}||m_{ee}^{(2)}|} = |Y_3 + Y_2 e^{2i\gamma} + Y_1 e^{-2i\beta}|^2. \quad (\text{A.5})$$

If we neglect Y_1 , we write $\gamma = \gamma_0$ and

$$\frac{|m_{ee}|^2}{|m_{ee}^{(3)}||m_{ee}^{(2)}|} = |Y_3 + Y_2 e^{2i\gamma_0}|^2, \quad (\text{A.6})$$

or

$$\cos(2\gamma_0) = \frac{1}{2} \left[\frac{|m_{ee}|^2}{|m_{ee}^{(3)}||m_{ee}^{(2)}|} - Y_3^2 - Y_2^2 \right]. \quad (\text{A.7})$$

Now, we will seek the γ by the perturbation with Y_1 , and expand γ as

$$\gamma = \sum_{n=0}^{\infty} \gamma_n Y_1^n = \gamma_0 + \gamma_1 Y_1 + \gamma_2 Y_1^2 + \dots. \quad (\text{A.8})$$

Then, it leads

$$\begin{aligned} \frac{|m_{ee}|^2}{|m_{ee}^{(3)}||m_{ee}^{(2)}|} &= |Y_3 + Y_2 e^{2i(\gamma_0 + \gamma_1 Y_1 + \gamma_2 Y_1^2 + \dots)} + Y_1 e^{-2i\beta}|^2 \\ &= |b_0 + b_1 Y_1 + b_2 Y_1^2 + \dots|^2, \end{aligned} \quad (\text{A.9})$$

where

$$\begin{aligned} b_0 &\equiv Y_3 + Y_2 e^{2i\gamma_0}, \\ b_1 &\equiv 2i\gamma_1 Y_2 e^{2i\gamma_0} + e^{-2i\beta}, \end{aligned} \quad (\text{A.10})$$

and so on. The b_0 satisfies

$$\frac{|m_{ee}|^2}{|m_{ee}^{(3)}||m_{ee}^{(2)}|} = |b_0|^2. \quad (\text{A.11})$$

When the order of Y_1^2 is neglected, one has

$$\frac{|m_{ee}|^2}{|m_{ee}^{(3)}||m_{ee}^{(2)}|} = |b_0 + b_1 Y_1 + O(Y_1^2)|^2 = |b_0|^2 + 2\text{Re}(b_0^* b_1) Y_1 + O(Y_1^2). \quad (\text{A.12})$$

The γ_1 can be determined by

$$\text{Re}(b_0^* b_1) = 0, \quad (\text{A.13})$$

and the result is

$$\begin{aligned} \gamma_1 &= \frac{1}{2 \sin(2\gamma_0)} \{(\text{Re } b_0) \cos(-2\beta) + (\text{Im } b_0) \sin(-2\beta)\} \\ &= \frac{1}{2 \sin(2\gamma_0)} \{|b_0| \cos(-2\beta + \varphi)\} \\ &= \frac{1}{2 \sin(2\gamma_0)} \sqrt{Y_3^2 + Y_2^2 + 2 \cos(2\gamma_0)} \cos(-2\beta + \varphi) \\ &= \frac{1}{2 \sin(2\gamma_0)} \frac{|m_{ee}|}{\sqrt{|m_{ee}^{(3)}||m_{ee}^{(2)}|}} \cos(-2\beta + \varphi), \end{aligned} \quad (\text{A.14})$$

where

$$\tan \varphi = -\frac{\text{Im } b_0}{\text{Re } b_0} = -\frac{Y_2 \sin(2\gamma_0)}{Y_3 + Y_2 \cos(2\gamma_0)}. \quad (\text{A.15})$$

Since $\alpha - \beta = \gamma = \gamma_0 + \gamma_1 Y_1 + O(Y_1^2)$, we have

$$\begin{aligned} \alpha &= \gamma_0 + \beta + Y_1 \frac{1}{2 \sin(2\gamma_0)} \sqrt{Y_3^2 + Y_2^2 + 2 \cos(2\gamma_0)} \cos(2\beta - \varphi) + O(Y_1^2) \\ &= \gamma_0 + \beta + \left(\frac{m_1}{m_3 m_2} \right) \frac{|m_{ee}|}{2 \sin(2\gamma_0)} \frac{|U_{e1}|^2}{|U_{e3}|^2 |U_{e2}|^2} \cos(2\beta - \varphi) + O(Y_1^2). \end{aligned} \quad (\text{A.16})$$

In this expansion, it should be $\sin 2\gamma_0 \neq 0$. This shows that we can not use this expansion near $\gamma_0 = 0$ or $\gamma_0 = -\pi/2$. When $\gamma_0 \approx 0$, it becomes $\cos(2\gamma_0) \approx 1$, and $|m_{ee}| \approx |m_{ee}^{(2)}| + |m_{ee}^{(3)}| \approx |m_{ee}|_{\text{max}}$. When $\gamma_0 \approx -\pi/2$, it becomes $\cos(2\gamma_0) \approx -1$, and $|m_{ee}| \approx \sqrt{|m_{ee}^{(2)}|^2 + |m_{ee}^{(3)}|^2 - 2|m_{ee}^{(2)}||m_{ee}^{(3)}|} = |m_{ee}^{(2)}| - |m_{ee}^{(3)}| \approx |m_{ee}|_{\text{min}}$. From these, we understand that the expansion in Y_1 can not be used near the maximum value $|m_{ee}|_{\text{max}}$ or the minimum value $|m_{ee}|_{\text{min}}$.

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